Math 106, Practice exam.

Problem 1.

Let $u(x, y) = x^2 - y^2 - 2xy$ and $v(x, y) = x^2 - y^2 + 2xy$. Let $f(x+iy) = u(x, y) + iv(x, y)$. Prove that f is entire. Compute $f'(i)$.

Problem 2.

Find the principal value of

$$\left(\frac{e}{2}(-1-\sqrt{3}i)\right)^{3\pi i}.$$

Problem 3.

Let $u(x, y) = (e^y + e^{-y})\cos x + (e^y - e^{-y})\sin x$. Check that u is harmonic. Find an entire function $f(z)$ whose real part is $u(x, y)$, such that $f(0) = 2$.

Problem 4.

Let $u(x, y)$ and $v(x, y)$ be harmonic conjugates. Show that $u(x^2 - y^2, 2xy)$ and $v(x^2 - y^2, 2xy)$ are also harmonic conjugates.

Problem 5.

Let R be the closed rectangle of side 2 centered at the origin. Find the points of R where $|f(z)|$ is minimum/maximum, for $f(z) = 2z + z^2$.

Problem 6.

Let $\mathfrak{h}^+$ denote the half plane of complex numbers z with Re $z > 0$. Let Δ be the open unit disc centered at the origin.

(i) Show that the Cayley function

$$C(z) = \frac{z-1}{z+1}$$

is a holomorphic function in $\mathfrak{h}^+$, such that for all $z \in \mathfrak{h}^+$ we have $f(z) \in \Delta$.

(ii) Are there any nonconstant entire functions f such that $f(z) \in \Delta$ for all z?

(iii) Are there any nonconstant entire functions f such that $f(z) \in \mathfrak{h}^+$ for all z?

Problem 7.

Let $f(z) = \pi \exp(\pi \bar{z})$ and let C be the boundary of the square with vertices at 0, 1, i, $1+i$ with the counterclockwise orientation. Find $\int_C f(z)dz$.

Problem 8.

Find the following residues

(i) Res $_{z=\pi} \frac{z-\sin z}{z^2 \sin z}$

(ii) Res $_{z=-i} \frac{\sqrt{z}}{(z^2+1)^2}$

The principal value is used for the square root in the second function.

Problem 9.

Find the different Laurent expansions of the function

$$\frac{3}{z^2+5z+4}$$

in powers of z, and indicate the regions where they are valid.

Problem 10.

Let

$$f(z) = \frac{1}{z(z-a)(z-b)}.$$

It is assumed that $0 < |a| < |b| < r < R$ are fixed. Let C be the counterclockwise circle of radius R centered at the origin.

(i) Find the singularities of f, indicate their type, and compute their residues.
(ii) Show that $\int_C f(z)dz = 0$.
(iii) Now consider the Laurent expansion of the function $f(z)$ in the region $r < |z| < \infty$. Write down the first two-three terms appearing in the Laurent expansion. One way to answer this question is to multiply together the individual Laurent expansions of $\frac{1}{z}$, $\frac{1}{z-a}$ and $\frac{1}{z-b}$. What is the coefficient of z^{-1} in this Laurent expansion? Does this contradict the answer that you found in (i) for the residue of f at 0?
(iv) Integrate the Laurent expansion term by term along C and confirm the answer you found in (ii).
(v) Give an example of a simple closed path C for which the answer in (ii) becomes nonzero.
(vi) Now generalize! Assume P is any polynomial of degree at most $n-2$ and let $a_1, \dots, a_n$ be arbitrary complex numbers. Assume R is large enough, in particular that the circle of radius R around the origin encloses all $a_1, \dots, a_n$. Show that

$$\int_C \frac{P(z)}{(z-a_1)\dots(z-a_n)}dz = 0.$$

Problem 11.

Compute the following two integrals

(i) $\int_0^\infty \frac{x^2dx}{(x^2+1)(x^2+4)}$.
(ii) $\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}$ with $a, b > 0$ real numbers.

Problem 12.

Let R be the square of side 2 centered at the origin and let $f(z) = e^z - 3z^3$. How many zeroes does f have in the interior of R? Show that they are simple zeroes.