Stanford University
Educational Program for Gifted Youth (EPGY)
Math Olympiad Problem Solving
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Contents

1 Invariants

An **invariant** is some aspect of a problem–typically a numerical quantity–that does not change, even if many other properties do change. You can use often use invariants to simplify difficult problems. Some examples of invariants are **parity**, **divisibility**, and **symmetry**.

Example 1.1 Two diametrically opposite corners of a chess board are deleted. Is it possible to tile the remaining 62 squares with 31 dominos?

Solution: No. Each domino covers one red square and one black square. But diametrically opposite corners are of the same color, so such a tiling is impossible.

Example 1.2 The numbers $1, 2, \ldots, 10$ are written in a row. Show that no matter what choice of sign $\pm$ is put in between them, the sum will never be 0.

Solution: The sum $1+2+\cdots+10 = 55$, an odd integer. Since parity is not affected by the choice of sign, for any choice of sign $\pm 1 \pm 2 \pm \cdots \pm 10$ will never be even, so in particular it will never be 0.

Example 1.3 All the dominos in a set are laid out in a chain according to the rules of the game. If one end of the chain is a 6, what is at the other end?

Solution: At the other end there must also be a 6. Each number of spots must occur in a pair, so that we may put them end to end. Since there are eight 6's, this last 6 pairs off with the one at the beginning of the chain.

Example 1.4 Let $a_1, a_2, \ldots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, 3, \ldots, n$. Prove that, if n is odd, then the product

$$(a_1 - 1)(a_2 - 2)(a_3 - 3)\cdots(a_n - n)$$

is even.

Solution: Consider the *sum* of the terms:

$$\begin{aligned}(a_1 - 1) + (a_2 - 2) + \cdots (a_n - n) &= (a_1 + a_2 + \cdots a_n) - (1 + 2 + \cdots + n)\\ &= (1 + 2 + \cdots + n) - (1 + 2 + \cdots + n)\\ &= 0.\end{aligned}$$

Thus, the sum of the terms is zero no matter what the arrangement. A sum of an odd number of integers which equals zero (an even number) must contain one even number, and any product that contains one even number is even.

Example 1.5 (Gaussian Pairing) Find the sum of the first 100 positive integers.

Solution:

$$\begin{aligned} S &= 1+2+3+\cdots+98+99+100 \\ S &= 100+99+98+\cdots+3+2+1 \\ 2S &= 100\cdot 101 \end{aligned}$$

Thus, $S = 50 \cdot 101 = 5050$.

Example 1.6 Let $d(n)$ denote the number of divisors of a positive integer n. Show that $d(n)$ is odd if and only if n is a perfect square.

Solution: The symmetry here is that we can always pair a divisor d of n with n/d. For example, if $n = 28$, it is natural to pair the divisor 2 with the divisor 14. Thus, as we go through the list of divisors of n, each divisor will have a unique partner, *unless n is a perfect square*, in which case $\sqrt{n}$ is paired with itself. For example, the divisors of 28 are 1, 2, 4, 7, 14, 28, which can be arranged in the pairs $(1, 28), (2, 14), (4, 7)$, so clearly $d(28)$ is even. On the other hand, the divisors of the perfect square 36 are $(1, 36), (2, 18), (3, 12), (4, 9), 6$. Notice that 6 has no partner, so $d(6)$ is odd. We conclude that $d(n)$ is odd if and only if n is a perfect square.

Example 1.7 Solve $x^4 + x^3 + x^2 + x + 1 = 0$.

Solution: We can use the symmetry of the coefficients to approach this problem. First, we divide by x^2 to obtain:

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0.$$

Note that this produces additional symmetry, as we can collect "like" terms as follows:

$$x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0.$$

Next, we make the substitution

$$u = x + \frac{1}{x}.$$

Then

$$u^2 = x^2 + 2 + \frac{1}{x^2},$$

so the equation becomes

$$u^2 - 2 + u + 1 = 0,$$

or

$$u^2 + u - 1 = 0.$$

Then

$$u = \frac{-1 \pm \sqrt{5}}{2},$$

and

$$x = \frac{u \pm \sqrt{u^2 - 4}}{2} = \frac{-1 \pm \sqrt{5} \pm i\sqrt{2\sqrt{5} \pm 10}}{4}.$$

Example 1.8 At first, a room is empty. Each minute, either one person enters or two people leave. After exactly 3^{1999} minutes, could the room contain $3^{1000} + 2$ people?

Solution: If there are n people in the room, after one minute, there will be either $n+1$ or $n-2$ people. The difference between these two possible outcomes is 3. After two minutes, there will be either $n-1$, $n+2$, or $n-4$ people. These possible values differ from one another by multiples of 3. Continuing for longer times, we see that at any fixed time t, the possible values for the population of the room differ from one another by multiples of 3. After 3^{1999} minutes, one possible population is 3^{1999} (this population occurs if one person entered each minute). This is a multiple of 3, so all the possible populations of the room at this time must also be a multiple of 3. Thus, $3^{1000} + 2$ is not a possible population after 3^{1999} minutes.

Example 1.9 A rectangle is tiled with smaller rectangles, each of which has at least one side of integral length. Prove that the tiled rectangle must also have at least one side of integral length.

Solution: Note: there are at least fourteen different solutions of this problem, several of which use invariants in different ways. See Stan Wagon, *Fourteen proofs of a result about tiling a rectangle*, American Mathematical Monthly, 94:601–617, 1987 for an account of these solutions.

For ease of discussion and notation, let's call the property of having at least one integral side "good." We must show that the large rectangle is "good." Start by orienting the large rectangle so that the lower-left corner is a lattice point. A *lattice point* (m, n) on the plane is one having integer coordinates. Next, we make the following key observation: *If the rectangle weren't good, then it would have only one lattice point corner. But if the rectangle is good, then it will have either two lattice point corners (if one dimension is an integer), or four lattice point corners (if both length and width are integers).*

Thus, the rectangle is good if and only if the number of corner lattice points is even. We'll count lattice point corners, and try to show that the number of lattice point corners of the big rectangle must be even.

Consider the corners of a small rectangle. It may have zero lattice point corners, or two lattice point corners, or four lattice point corners, but it can not have just one or three, because each small rectangle is good (i.e. has at least one integer-length side). Thus, if we count the number of corner lattice points on each small rectangle and add them up, the sum, which we call S, must be even.

Note, however, that we have overcounted some of the lattice points–in particular, we have overcounted any lattice points that occur as the corner of *more than one* rectangle. We make the following observations:

- We will only count a corner lattice point once, twice, or four times–never three times.
- The only corner lattice points that are counted exactly once are the corners of the large rectangle.

Let a denote the number of corner lattice points that we count exactly once (i.e. the corner lattice points of the large rectangle), let b denote the number of corner lattice points that we count exactly twice, and let c denote the number of corner lattice points that we count exactly four times. Then we have:

$$S = a + 2b + 4c,$$

so

$$a = S - 2b - 4c.$$

Since S, $2b$, and $4c$ are all even, a is even as well. Thus, the large rectangle has an even number of corner lattice points, so it must have two or four corner lattice points. Thus, the large rectangle has at least one side of integral length.

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2 Arithmetic Ratios

Example 2.1 (2000 AMC 10 #8) At Olympic High School, 2/5 of the freshmen and 4/5 of the sophomores took the AMC 10. The number of freshmen and sophomore contestants was the same. Which of the following must be true?

(A) There are five times as many sophomores as freshmen.

(B) There are twice as many sophomores as freshmen.

(C) There are as many sophomores as freshmen.

(D) There are twice as many freshmen as sophomores.

(E) There are five times as many freshmen as sophomores.

Solution: Let's arbitrarily *assume* that there are 100 freshmen in the school. Then 40 freshmen and 40 sophomores took the exam. Thus, there are 50 sophomores in the school, so the answer is (D). Note: it was convenient to assign a specific value for the number of freshmen, but it is *not* necessary. We could let F denote the number of freshmen and conclude, using the same technique as above, that there are $F/2$ sophomores. We chose to use 100 for the number of freshmen because it simplifies the situation.

Example 2.2 (2004 AMC 10A #17 and 2004 AMC 12A #15) Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. Each girl runs at a constant speed. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. What is the length of the track in meters?

(A) 250 (B) 300 (C) 350 (D) 400 (E) 500

Solution: The fundamental relationship involved in this problem is

$$\text{rate} = \frac{\text{distance}}{\text{time}}.$$

First we must define some notation. Let

- L be the length of the track.
- R_S be the rate at which Sally runs.
- R_B be the rate at which Brenda runs.
- T_1 be the time it takes them to first meet.
- T_2 be the time after they first meet until they meet again.

Solution. They start at opposite sides of the track and run in opposite directions, so they first meet when their combined distance run is $L/2$. We are told that Brenda has run 100 meters during this time T_1, so

$$100 = T_1 \cdot R_B.$$

When they meet again they have together run the full length, L, of the track since their first meeting. Since their speeds are constant and they ran together $L/2$ in time T_1, we have $T_2 = 2T_1$. Sally has run 150 meters during the time T_2, so

$$L = T_2 \cdot R_S + T_2 \cdot R_B = 150 + 2T_1 \cdot R_B = 150 + 2 \cdot 100 = 350 \text{ meters}.$$

Example 2.3 (2002 AMC 10B #21 and 2002 AMC 12B #17) Andy's lawn has twice as much area as Beth's lawn and three times as much area as Carlos' lawn. Carlos' lawn mower cuts half as fast as Beth's mower and one third as fast as Andy's mower. They all start to mow their lawns at the same time. Who will finish first?

(A) Andy

(B) Beth

(C) Carlos

(D) Andy and Carlos tie for first.

(E) They all tie.

Solution. This is also a rate problem, so we begin with some notation. Let

- $A_a,\ R_a,\ T_a$ be the area, rate, and time for Andy.
- $A_b,\ R_b,\ T_b$ be the area, rate, and time for Beth.
- $A_c,\ R_c,\ T_c$ be the area, rate, and time for Carlos.

Then $A_b = A_a/2$, $A_c = A_a/3$, $R_b = 2R_c$, and $R_a = 3R_c$. To solve the problem, we must express the times for each worker using a common base. Here, we will use the fraction A_a/R_c, but any combination of A/R is possible. We obtain:

- $T_a = \frac{1}{3}\frac{A_a}{R_c}$
- $T_b = \frac{1}{4}\frac{A_a}{R_c}$
- $T_c = \frac{1}{3}\frac{A_a}{R_c}$

Thus, Beth finishes first.

Example 2.4 (1988 AHSME #8) If $\dfrac{b}{a} = 2$ and $\dfrac{c}{b} = 3$, what is the ratio of $a+b$ to $b+c$?

(A) $\frac{1}{3}$ (B) $\frac{3}{8}$ (C) $\frac{3}{5}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

Solution. Assume that $a = 3$ and $b = 6$. Then $c = 18$, so

$$\frac{a+b}{b+c} = \frac{9}{24} = \frac{3}{8}.$$

Example 2.5 (1983 AHSME #7) Alice sells an item at \$10 less than the list price and receives 10% of her selling price as her commission. Bob sells the same item at \$20 less than the list price and receives 20% of his selling price as his commission. If they both get the same commission, then the list price is

(A) \$20 (B) \$30 (C) \$50 (D) \$70 (E) \$100

Solution. Let p denote the list price of the item (in dollars). Alice's commission is

$$0.10 \cdot (p - 10),$$

and Bob's commission is

$$0.20 \cdot (p - 20).$$

Thus,

$$0.10 \cdot (p - 10) = 0.20 \cdot (p - 20),$$

so

$$p = 30.$$

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3 Algebraic Manipulation

Example 3.1 If $x + y = xy = 3$, find $x^3 + y^3$.

Solution: One way to solve this problem would be to solve the system

$$x + y = xy = 3$$

for x and y, and then substitute the resulting values into the expression x^3+y^3. This technique would certainly work, but would be time-consuming, messy, and probably error-prone. Instead, we'll use algebra to construct a more elegant solution. Since our goal is to find $x^3 + y^3$, we'll start by trying to find $x^2 + y^2$. We have:

$$\begin{aligned} 3^2 &= (x+y)^2 \\ &= x^2 + 2xy + y^2 \\ &= x^2 + y^2 + 2 \cdot 3 \\ &= x^2 + y^2 + 6. \end{aligned}$$

Thus, $x^2 + y^2 = 3^2 - 6 = 3$. Next, we obtain:

$$\begin{aligned} 3 \cdot 3 &= (x+y)(x^2+y^2) \\ &= x^3 + y^3 + x^2y + xy^2 + y^3 \\ &= x^3 + y^3 + xy(x+y) \\ &= x^3 + y^3 + 3 \cdot 3. \end{aligned}$$

Finally, we conclude that

$$x^3 + y^3 = 3 \cdot 3 - 3 \cdot 3 = 0.$$

Example 3.2 Find positive integers a and b with

$$\sqrt{5 + \sqrt{24}} = \sqrt{a} + \sqrt{b}.$$

Solution: Observe that

$$5 + \sqrt{24} = 3 + 2\sqrt{2 \cdot 3} + 2 = (\sqrt{2} + \sqrt{3})^2.$$

Thus,

$$\sqrt{5 + 2\sqrt{6}} = \sqrt{2} + \sqrt{3}.$$

You should know (and actively use) the following factorization formulas:

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x-y)^2 = x^2 - 2xy + y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x+y)$
- $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 = x^3 - y^3 - 3xy(x-y)$
- $x^2 - y^2 = (x+y)(x-y)$
- $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$ for all positive integers n
- $x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - xy^{n-2} + y^{n-1})$ for all positive odd integers n (the terms of the second factor alternate in sign)
- $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$
- $(x+y+z+w)^2 = x^2 + y^2 + z^2 + w^2 + 2xy + 2xz + 2xw + 2yz + 2yw + 2zw$
- $(x-y)^2 + 4xy = (x+y)^2$

3.1 Single-variable Equations

Example 3.3 Solve the equation $|x-3|^{(x^2-8x+15)/(x-2)} = 1$.

Solution: If
$$a^b = 1,$$
the either $a = 1$ or $b = 0$. Note that 0^0 is undefined, so we cannot have $a = b = 0$. Thus, we start by setting $|x-3| = 1$. This implies
$$x = 4 \text{ or } x = 2.$$
We discard $x = 2$ as the exponent is undefined at $x = 2$. Next, we set $x^2 - 8x + 15 = 0$, which implies
$$x = 5 \text{ or } x = 3.$$
We discard $x = 3$ since this would give 0^0. Thus, the only solutions are $x = 4$ and $x = 5$.

Example 3.4 Solve $9 + x^{-4} = 10x^{-2}$.

Solution: First, we rewrite the equation as
$$x^{-4} - 10x^{-2} + 9 = 0.$$
Observe that
$$x^{-4} - 10x^{-2} + 9 = (x^{-2} - 9)(x^{-2} - 1).$$
Thus the solutions are
$$x = \pm\frac{1}{3}, \quad \pm 1.$$

Example 3.5 Solve

$$(x-5)(x-7)(x+6)(x+4) = 504.$$

Solution: Reorder the factors and multiply to obtain

$$(x-5)(x-7)(x+6)(x+4) = (x-5)(x+4)(x-7)(x+6) = (x^2-x-20)(x^2-x-42).$$

Set $y = x^2 - x$. Then:

$$\begin{aligned} (y-20)(y-42) &= 504 \\ y^2 - 62y + 336 &= (y-6)(y-56) = 0. \end{aligned}$$

Thus we obtain $y = 6, 56$, so

$$x^2 - x = 6$$

and

$$x^2 - x = 56.$$

Solving both quadratics, we obtain

$$x = -2, 4, -7, 8.$$

Example 3.6 Solve $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.

Solution: Reordering, we obtain:

$$\begin{aligned} 12x^4 + 12 - 56(x^3 + x) + 89x^2 &= 0 \\ x^2 \left[12(x^2 + \frac{1}{x^2}) - 56(x + \frac{1}{x}) + 89 \right] &= 0 \\ 12(x^2 + \frac{1}{x^2}) - 56(x + \frac{1}{x}) + 89 &= 0. \end{aligned}$$

Next, we set $u = x + 1/x$. Then

$$u^2 - 2 = x^2 + 1/x^2,$$

Using this, we obtain

$$12(u^2 - 2) - 56u + 89 = 0,$$

so $u = 5/2,\ 13/6$. Thus,

$$x + \frac{1}{x} = \frac{5}{2}$$

and

$$x + \frac{1}{x} = \frac{13}{6}.$$

Finally, solving both equations above, we conclude that

$$x = 1/2, 2, 2/3, 3/2.$$

Example 3.7 Find the real solutions to

$$x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12.$$

Solution: Observe that we can rewrite the given equation as

$$x^2 - 5x + 3 + 2\sqrt{x^2 - 5x + 3} - 15 = 0.$$

Next, set $u = x^2 - 5x + 3$. Then:

$$\begin{aligned} u + 2u^{1/2} - 15 &= 0 \\ (u^{1/2} + 5)(u^{1/2} - 3) &= 0. \end{aligned}$$

Thus $u = 9$, and $x^2 - 5x + 3 = 9$. Finally, we conclude that

$$x = -1,\ 6.$$

Example 3.8 Solve

$$\sqrt[3]{14 + x} + \sqrt[3]{14 - x} = 4.$$

Solution: Let $u = \sqrt[3]{14 + x}$ and $v = \sqrt[3]{14 - x}$. Then

$$\begin{aligned} 64 &= (u + v)^3 \\ &= u^3 + v^3 + 3uv(u + v) \\ &= 14 + x + 14 - x + 12(196 - x^2)^{1/3}. \end{aligned}$$

Thus,

$$3 = (196 - x^2)^{1/3},$$

which we solve to obtain $x = \pm 13$.

Example 3.9 Solve $9^x - 3^{x+1} - 4 = 0$.

Solution: Observe that $9^x - 3^{x+1} - 4 = (3^x - 4)(3^x + 1)$. Since no real number x satisfies $3^x + 1 = 0$, we discard this factor. Thus, $3^x - 4 = 0$, and we obtain $x = \log_3 4$.

Example 3.10 Find the exact value of $\cos 2\pi/5$.

Solution: Using the identity

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

twice, we obtain

$$\cos 2\theta = 2\cos^2\theta - 1$$

and

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Let $x = \cos 2\pi/5$. Since $\cos 6\pi/5 = \cos 4\pi/5$, we see that x satisfies the equation

$$4x^3 - 2x^2 - 3x + 1 = 0,$$

which we can rewrite as

$$(x-1)(4x^2 + 2x - 1) = 0.$$

Since $x = \cos 2\pi/5 \neq 1$, and $\cos 2\pi/5 > 0$, x is the positive root of the quadratic equation $4x^2 + 2x - 1 = 0$. We conclude that

$$x = \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}.$$

Example 3.11 Solve the equation $2^{|x|} = \sin x^2$.

Solution: Clearly $x = 0$ is not a solution. Since $2^y > 1$ for $y > 0$, and $1 \leq \sin y \leq 1$ for all y, the equation does not have a solution.

Example 3.12 How many real numbers x satisfy

$$\sin x = \frac{x}{100}?$$

Solution: Clearly, $x = 0$ is a solution. Also, if $x > 0$ is a solution, so is $-x < 0$. So, we can restrict ourselves to positive solutions.

If x is a solution then $|x| = 100|\sin x| \leq 100$. So we can further restrict x to the interval $(0, 100]$. Decompose $(0, 100]$ into 2π-long intervals (the last interval is shorter):

$$(0, 100] = (0, 2\pi] \ \cup\ (2\pi, 4\pi] \ \cup\ (4\pi, 6\pi] \ \cup\ \cdots \ \cup\ (28\pi, 30\pi] \ \cup\ (30\pi, 100].$$

From the graphs of $y = \sin x$ and $y = x/100$ we that the interval $(0, 2\pi]$ contains only one solution. Each interval of the form $(2\pi k,\ 2(k+1)\pi], k = 1, 2, \ldots, 14$ contains two solutions. Since $31\pi < 100$, the interval $]30\pi; 100]$ contains a full wave, so it contains two solutions. Thus, there are $1 + 2 \cdot 14 + 2 = 31$ positive solutions, and hence, 31 negative solutions. Therefore, there is a total of $31 + 31 + 1 = 63$ solutions.

3.2 Systems of Equations

Example 3.13 Solve the system of equations

$$\begin{aligned} x + y + u &= 4, \\ y + u + v &= -5, \\ u + v + x &= 0, \\ v + x + y &= -8. \end{aligned}$$

Solution: Adding all the equations and dividing by 3, we obtain

$$x + y + u + v = -3.$$

This implies that

$$\begin{aligned} 4 + v &= -3, \\ -5 + x &= -3, \\ 0 + y &= -3, \\ -8 + u &= -3. \end{aligned}$$

Thus

$$x = 2, y = -3, u = 5, v = -7.$$

Example 3.14 Solve the system

$$(x + y)(x + z) = 30,$$

$$(y + z)(y + x) = 15,$$

$$(z + x)(z + y) = 18.$$

Solution: Set $u = y + z, v = z + x, w = x + y$. Then the system becomes

$$\begin{aligned} vw &= 30 \\ wu &= 15 \\ uv &= 18. \end{aligned}$$

Multiplying these equations we obtain

$$u^2v^2w^2 = 8100,$$

which we solve to obtain $uvw = \pm 90$. Next, we solve for u, v, and w to obtain $u = 3, v = 6, w = 5$, or $u = -3, v = -6, w = -5$. Then we have:

$$\begin{aligned} y + z &= \pm 3 \\ z + x &= \pm 6 \\ x + y &= \pm 5. \end{aligned}$$

Thus we conclude that

$$x = 4, y = 1, z = 2 \text{ or } x = -4, y = -1, z = -2.$$

4 Polynomials

Theorem 4.1 The **Division Algorithm** for polynomials. If the polynomial $p(x)$ is divided by $d(x)$ then there exist polynomials $q(x), r(x)$ such that

$$p(x) = d(x)q(x) + r(x)$$

and $0 \leq \text{degree}(r(x)) < \text{degree}(d(x))$.

We can find the quotient $q(x)$ and the remainder $r(x)$ by performing ordinary long division with polynomials. For example, if $x^5 + x^4 + 1$ is divided by $x^2 + 1$ we obtain

$$x^5 + x^4 + 1 = (x^3 + x^2 - x - 1)(x^2 + 1) + x + 2,$$

so the quotient is $q(x) = x^3 + x^2 - x - 1$ and the remainder is $r(x) = x + 2$.

Theorem 4.2 The **Remainder Theorem** for polynomials. If the polynomial $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

Proof. By the Division Algorithm for polynomials, we know that

$$p(x) = (x - a)q(x) + r.$$

Note that the remainder $r(x) = r$ is a constant (degree 0) since the degree of $x - a$ is equal to 1. Substituting $x = a$, we obtain

$$p(a) = (a - a)q(a) + r = 0 + r = r.$$

Theorem 4.3 The **Factor Theorem** for polynomials. The polynomial $p(x)$ is divisible by $x - a$ if and only if $p(a) = 0$.

Proof. The remainder r is equal to $p(a)$, so $p(x)$ is divisible by $x - a$ if and only if $p(a) = 0$.

Example 4.1 Find the remainder when $(x + 3)^5 + (x + 2)^8 + (5x + 9)^{1997}$ is divided by $x + 2$.

Solution: Let $p(x)$ be the polynomial $(x+3)^5+(x+2)^8+(5x+9)^{1997}$. The remainder upon dividing $P(x)$ by $x + 2 = x - (-2)$ is $p(-2) = 0$.

Example 4.2 A polynomial $p(x)$ leaves remainder -2 upon division by $x - 1$ and remainder -4 upon division by $x + 2$. Find the remainder when this polynomial is divided by $x^2 + x - 2$.

Solution: There exist polynomials $q_1(x)$ and $q_2(x)$ such that $p(x) = q_1(x)(x-1)-2$ and $p(x) = q_2(x)(x + 2) - 4$. Thus

$$p(1) = -2 \text{ and } p(-2) = -4.$$

Since

$$x^2 + x - 2 = (x - 1)(x + 2)$$

is a polynomial of degree 2, the remainder $r(x)$ upon dividing $p(x)$ by $x^2 + x - 1$ is of degree 1 or less, so $r(x) = ax + b$ for some constants a and b which we must determine. By the Division Algorithm for polynomials,

$$p(x) = q(x)(x^2 + x - 1) + ax + b.$$

Thus, we have

$$-2 = p(1) = a + b$$

and

$$-4 = p(-2) = -2a + b.$$

We solve this system to obtain $a = 2/3$ and $b = -8/3$. The desired remainder is thus

$$r(x) = \frac{2}{3}x - \frac{8}{3}.$$

Example 4.3 Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Find the remainder when $f(x^5)$ is divided by $f(x)$.

Solution: Observe that $f(x)(x - 1) = x^5 - 1$ and

$$f(x^5) = x^{20} + x^{15} + x^{10} + x^5 + 1 = (x^{20} - 1) + (x^{15} - 1) + (x^{10} - 1) + (x^5 - 1) + 5.$$

Each of the summands in parentheses is divisible by $x^5 - 1$ and by $f(x)$. The remainder is thus 5.

Example 4.4 If $p(x)$ is a cubic polynomial with $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$, find $p(6)$.

Solution: Let

$$g(x) = p(x) - x.$$

Observe that $g(x)$ is a polynomial of degree 3 and that

$$g(1) = g(2) = g(3) = 0.$$

Thus

$$g(x) = c(x - 1)(x - 2)(x - 3)$$

for some constant c that we must determine. Now,

$$g(4) = c(4 - 1)(4 - 2)(4 - 3) = 6c$$

and

$$g(4) = p(4) - 4 = 1,$$

so

$$c = 1/6.$$

Finally,

$$p(6) = g(6) + 6 = \frac{(6-1)(6-2)(6-3)}{6} + 6 = 16.$$

Theorem 4.4 The Rational Roots (Zeros) Test: Suppose that $a_0,\ a_1, \ldots, a_n$ are integers with $a_n \neq 0$. If p/q is a rational root (zero) of

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

then p divides a_0 and q divides a_1.

Next, we will consider the **relationship between the zeros of a polynomial and the coefficients of the polynomial**.

Quadratic polynomials. We'll begin with the case of a quadratic (monic) polynomial. Suppose that the zeros of the quadratic polynomial $x^2 + a_1 x + a_0$ are r and s. Then we can write the polynomial as

$$\begin{aligned} x^2 + a_1 x + a_0 &= (x-r)(x-s) \\ &= x^2 - (r+s)x + rs. \end{aligned}$$

Equating terms, we obtain

$$a_1 = -(r+s) \text{ and } a_0 = rs.$$

Thus, we conclude the following:

- $a_1 = -(\text{sum of the zeros})$
- $a_0 = +(\text{product of the zeros})$

Cubic Polynomials. Suppose that the zeros of the cubic polynomial $x^3 + a_2 x^2 + a_1 x + a_0$ are q, r, s. Then we can factor the cubic polynomial as

$$\begin{aligned} x^3 + a_2 x^2 + a_1 x + a_0 &= (x-q)(x-r)(x-s) \\ &= x^3 - (q+r+s)x^2 + (qr+qs+rs)x - qrs. \end{aligned}$$

Equating terms, we obtain

$$a_2 = -(q+r+s),\ a_1 = qr+qs+rs,\ a_0 = -qrs.$$

Thus, we conclude the following:

- $a_2 = -(\text{sum of the zeros})$
- $a_1 = +(\text{sum of all products of two different zeros})$
- $a_0 = -(\text{product of the zeros})$

In general, we have the following. Suppose that

$$r_1, r_2, \ldots, r_n$$

are the roots of the monic degree-n polynomial

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0.$$

Then, for $k = 1, 2, \ldots n$,

$$a_k = (-1)^{n-k}(\text{sum of all products of } n-k \text{ different zeros}).$$

Example 4.5 (1983 AIME) What is the product of the real roots of the equation

$$x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}?$$

Solution: The only obstacle to an immediate solution of this problem is the presence of the square root. Thus, a natural technique is to substitute

$$y = \sqrt{x^2 + 18x + 45}.$$

Note that if x is real, then y must be non-negative (i.e. $y \geq 0$). Then the equation becomes:

$$\begin{aligned} y^2 - 15 &= 2y \\ (y-5)(y+3) &= 0. \end{aligned}$$

Thus, $y = 5$ or $y = -3$. We reject $y = -3$ since $y \geq 0$. Substituting $y = 5$, we obtain

$$x^2 + 18x + 20 = 0.$$

The product of the roots of this quadratic polynomial is 20.

Example 4.6 (1984 USAMO) The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.

Solution: Let the zeros of the quartic be a, b, c, d. Then we have:

$$\begin{aligned} a+b+c+d &= 18 \\ ab+ac+ad+bc+bd+cd &= k \\ abc+abd+acd+bcd &= -200 \\ abcd &= -1994. \end{aligned}$$

Without loss of generality, let $ab = -32$. Substituting this into $abcd = -1984$, we obtain $cd = 62$. Then the equations become:

$$\begin{aligned} a+b+c+d &= 18 \\ 30+ac+ad+bc+bd &= k \\ -32c-32d+62a+62b &= -200 \end{aligned}$$

Recall that we need to compute k, *not* the values a, b, c, d. Note that if we could compute $ac+ad+bc+bd$, then we could compute k. We see that $ac+ad+bc+bd$ factors as

$$ac+ad+bc+bd = a(c+d)+b(c+d) = (a+b)(c+d).$$

Further,

$$-32c-32d+62a+62b = -32(c+d)+62(a+b).$$

Next, let $u = a+b$ and $v = c+d$. Then we have:

$$\begin{aligned} u+v &= 18 \\ 62u-32v &= -200. \end{aligned}$$

Solving, we obtain $u = 4$ and $v = 18$, so

$$k = 30 + 4 \cdot 14 = 86.$$

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5 Functions and Binary Operations

Definition 5.1 A **binary operation** on a set of numbers is a way to take two of the numbers and produce a third.

Often, a binary operation in a mathematics competition problem will be expressed using some unusual symbol, such as $\heartsuit$ or §. The result of the operation after it is applied to the numbers a and b would typically be written as $\heartsuit(a, b)$ or $a\heartsuit b$, or $\S(a, b)$ or $a\S b$.

Definition 5.2 A **function** f is a means of associating each element of one set, called the **domain** of f, with exactly one element of a second set, called the **range** of f.

Definition 5.3 The **floor function** $f(x) = \lfloor x \rfloor$ is defined to be the greatest integer less than or equal to x.

For example, $\lfloor 3.7 \rfloor = 3$, $\lfloor 2 \rfloor = 2$, $\lfloor -2.4 \rfloor = -3$.

Definition 5.4 The **ceiling function** $f(x) = \lceil x \rceil$ is defined to be the smallest integer greater than or equal to x.

For example, $\lceil 3.1 \rceil = 4$, $\lceil -1.2 \rceil = -1$.

Example 5.1 (1993 AHSME #4) Define the operation $\heartsuit$ by

$$x\heartsuit y = 4x - 3y + xy$$

for all real numbers x and y. For how many real numbers y does $12 = 3\heartsuit y$?

(A) 0 (B) 1 (C) 3 (D) 4 (E) more than 4

Solution:

$$\begin{aligned} 12 &= 3\heartsuit y \\ &= 4 \cdot 3 - 3 \cdot y + 3 \cdot y \\ &= 12. \end{aligned}$$

Thus, the equation is true for *any real number* y.

Example 5.2 (1996 AHSME #14) The function $E(n)$ is defined for each positive integer n to be the sum of the even digits of n. For example, $E(5681) = 6 + 8 = 14$. What is the value of

$$E(1) + E(2) + \cdots + E(100)?$$

(A) 200 (B) 300 (C) 400 (D) 900 (E) 2250

Solution: Rater than determine the values of $E(n)$ individually, observe that the numbers

$$00, 01, 02, \ldots, 99$$

contain an equal number of each of the digits $0, 1, \ldots, 9$. There are a total of $2 \cdot 100 = 200$ of these digits, so there are $200/10 = 20$ of each digit. Thus,

$$E(0) + E(1) + E(2) + \cdots + E(99) = 20(0 + 2 + 4 + 6 + 8) = 400.$$

Since $E(0) = 0$ and $E(100) = 0$, the required sum is also 400.

Example 5.3 (1981 AHSME #17) The function f has the property that, for all nonzero real numbers,

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x.$$

How many nonzero real number solutions are there to the equation

$$f(x) = f(-x)?$$

(A) 0

(B) 1

(C) 2

(D) All nonzero real numbers

(E) Infinitely many, but not all, nonzero real numbers

Solution: We need to determine some way to eliminate the term $f(1/x)$ from the equation that describes $f(x)$. Once we do this, we will be able to compare $f(x)$ and $f(-x)$. The first technique to try is to substitute $1/x$ for x in the defining equation for $f(x)$. We obtain:

$$f\left(\frac{1}{x}\right) + 2f(x) = 3 \cdot \frac{1}{x},$$

so

$$f\left(\frac{1}{x}\right) = \frac{3}{x} - 2f(x).$$

Substituting this expression for $f(1/x)$ into the original equation produces

$$f(x) + 2\left(\frac{3}{x} - 2f(x)\right) = 3x.$$

Thus,

$$-3f(x) + \frac{6}{x} = 3x,$$

and we conclude that

$$f(x) = \frac{2 - x^2}{x}.$$

Finally, setting $f(x) = f(-x)$ for $x \neq 0$, we obtain

$$\frac{2 - x^2}{x} = \frac{2 - (-x)^2}{-x},$$

so

$$x = \pm\sqrt{2}.$$

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6 Combinatorics: The Product and Sum Rules

6.1 The Product Rule

Theorem 6.1 The Product Rule. Suppose that an experiment E can be performed in k stages: E_1 first, E_2 second, ..., E_k last. Suppose moreover that E_i can be done in n_i different ways, and that the number of ways of performing E_i is not influenced by any predecessors $E_1, E_2, \ldots, E_{i-1}$. Then E_1 **and** E_2 **and** ... **and** E_k can occur simultaneously in $n_1 n_2 \cdots n_k$ ways.

Example 6.1 In a group of 8 men and 9 women we can pick one man **and** one woman in $8 \cdot 9 = 72$ ways. Notice that we are choosing two persons.

Example 6.2 A red die and a blue die are tossed. In how many ways can they land?

Solution: If we view the outcomes as an ordered pair (r, b) then by the multiplication principle we have the $6 \cdot 6 = 36$ possible outcomes

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

The red die can land in any of 6 ways,

$$\boxed{6}\boxed{}$$

and the blue die may land in any of 6 ways

$$\boxed{6}\boxed{6}.$$

Example 6.3 A multiple-choice test consists of 20 questions, each one with 4 choices. There are 4 ways of answering the first question, 4 ways of answering the second question, etc. Thus, there are $4^{20} = 1099511627776$ ways of answering the exam.

Example 6.4 There are $9 \cdot 10 \cdot 10 = 900$ positive 3-digit integers:

$$100, 101, 102, \ldots, 998, 999.$$

Since the leftmost integer cannot be 0, there are only 9 choices $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for it,

$$\boxed{9}\boxed{}\boxed{}.$$

There are 10 choices for the second digit

$$\boxed{9}\boxed{10}\boxed{},$$

and 10 choices for the last digit

$$\begin{array}{|c|c|c|}\hline 9 & 10 & 10 \\ \hline\end{array}.$$

Example 6.5 There are $9 \cdot 10 \cdot 5 = 450$ even positive 3-digit integers:

$$100, 102, 104, \ldots, 996, 998.$$

Since the leftmost integer cannot be 0, there are only 9 choices $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for it,

$$\begin{array}{|c|c|c|}\hline 9 & & \\ \hline\end{array}.$$

There are 10 choices for the second digit

$$\begin{array}{|c|c|c|}\hline 9 & 10 & \\ \hline\end{array}.$$

Since the integer must be even, the last digit must be one of the 5 choices $\{0, 2, 4, 6, 8\}$

$$\begin{array}{|c|c|c|}\hline 9 & 10 & 5 \\ \hline\end{array}.$$

Definition 6.1 A *palindromic integer* or *palindrome* is a positive integer whose decimal expansion is symmetric and that is not divisible by 10. In other words, one reads the same integer backwards or forwards.

Note: A palindrome in common parlance, is a word or phrase that reads the same backwards to forwards. The Philadelphia street name *Camac* is a palindrome. So are the phrases (if we ignore punctuation) (a) “A man, a plan, a canal, Panama!” (b) “Sit on a potato pan!, Otis.” (c) “Able was I ere I saw Elba.” This last one is attributed to Napoleon, though it is doubtful that he knew enough English to form it.

Example 6.6 The following integers are all palindromes:

$$1, 8, 11, 99, 101, 131, 999, 1234321, 9987899.$$

Example 6.7 How many palindromes are there of 5 digits?
Solution: There are 9 ways of choosing the leftmost digit.

$$\begin{array}{|c|c|c|c|c|}\hline 9 & & & & \\ \hline\end{array}.$$

Once the leftmost digit is chosen, the last digit must be identical to it, so we have

$$\begin{array}{|c|c|c|c|c|}\hline 9 & & & & 1 \\ \hline\end{array}.$$

There are 10 choices for the second digit from the left

$$\begin{array}{|c|c|c|c|c|}\hline 9 & 10 & & & 1 \\ \hline\end{array}.$$

Once this digit is chosen, the second digit from the right must be identical to it, so we have only 1 choice for it,

9	10		1	1

.

Finally, there are 10 choices for the third digit from the right,

9	10	10	1	1

,

which give us 900 palindromes of 5-digits.

Example 6.8 How many palindromes of 5 digits are even?

Solution: A five digit even palindrome has the form $ABCBA$, where A belongs to $\{2, 4, 6, 8\}$, and B, C belong to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Thus there are 4 choices for the first digit, 10 for the second, and 10 for the third. Once these digits are chosen, the palindrome is completely determined. Therefore, there are $4 \times 10 \times 10 = 400$ even palindromes of 5 digits.

Example 6.9 How many positive divisors does 300 have?

Solution: We have $300 = 3 \cdot 2^2 5^2$. Thus every factor of 300 is of the form $3^a 2^b 5^c$, where $0 \leq a \leq 1$, $0 \leq b \leq 2$, and $0 \leq c \leq 2$. Thus there are 2 choices for a, 3 for b and 3 for c. This gives $2 \cdot 3 \cdot 3 = 18$ positive divisors.

We now prove that if a set A has n elements, then it has 2^n subsets. To motivate the proof, consider the set $\{a, b, c\}$. To each element we attach a binary code of length 3. We write 0 if a particular element is not in the set and 1 if it is. We then have the following associations:

$\varnothing \leftrightarrow 000,$	$\{a, b\} \leftrightarrow 110,$
$\{a\} \leftrightarrow 100,$	$\{a, c\} \leftrightarrow 101,$
$\{b\} \leftrightarrow 010,$	$\{b, c\} \leftrightarrow 011,$
$\{c\} \leftrightarrow 001,$	$\{a, b, c\} \leftrightarrow 111.$

Thus there is a one-to-one correspondence between the subsets of a finite set of 3 elements and binary sequences of length 3.

Theorem 6.2 Cardinality of the Power Set. Let A be a finite set with $|A|$ card $(A) = n$. Then A has 2^n subsets.

Proof. We attach a binary code to each element of the subset, 1 if the element is in the subset and 0 if the element is not in the subset. The total number of subsets is the total number of such binary codes, and there are 2^n in number.

6.2 The Sum Rule

Theorem 6.3 The Sum Rule. Let $E_1, E_2, \dots, E_k$, be pairwise mutually exclusive events. If E_i can occur in n_i ways, then either E_1 **or** E_2 **or**, ..., **or** E_k can occur in

$$n_1 + n_2 + \cdots n_k$$

ways.

Remark. Notice that the "**or**" here is exclusive.

Example 6.10 In a group of 8 men and 9 women we can pick one man **or** one woman in $8 + 9 = 17$ ways. Notice that we are choosing one person.

Example 6.11 There are five Golden retrievers, six Irish setters, and eight Poodles at the pound. How many ways can two dogs be chosen if they are not the same kind?

Solution: We choose: a Golden retriever **and** an Irish setter **or** a Golden retriever **and** a Poodle **or** an Irish setter **and** a Poodle.

One Golden retriever and one Irish setter can be chosen in $5 \cdot 6 = 30$ ways; one Golden retriever and one Poodle can be chosen in $5 \cdot 8 = 40$ ways; one Irish setter and one Poodle can be chosen in $6 \cdot 8 = 48$ ways. By the sum rule, there are $30 + 40 + 48 = 118$ combinations.

Example 6.12 There were a total of 1890 digits used in writing all of the page numbers in a book. How many pages does the book have?

Solution: A total of

$$1 \cdot 9 + 2 \cdot 90 = 189$$

digits are used to write pages 1 to 99, inclusive. We have of $1890 - 189 = 1701$ digits left, which is enough for $1701/3 = 567$ extra pages (starting from page 100). The book has $99 + 567 = 666$ pages.

Example 6.13 How many 4-digit integers can be formed with the set of digits $\{0, 1, 2, 3, 4, 5\}$ such that no digit is repeated and the resulting integer is a multiple of 3?

Solution: The integers desired have the form $D_1D_2D_3D_4$ with $D_1 \neq 0$. Under the stipulated constraints, we must have

$$D_1 + D_2 + D_3 + D_4 \in \{6, 9, 12\}.$$

We thus consider three cases.

Case I: $D_1+D_2+D_3+D_4=6$. Here we have $\{D_1,D_2,D_3,D_4\}=\{0,1,2,3,4\}$, $D_1\neq 0$. There are then 3 choices for D_1. After D_1 is chosen, D_2 can be chosen in 3 ways, D_3 in 2 ways, and D_1 in 1 way. There are thus $3\times 3\times 2\times 1=3\cdot 3!=18$ integers satisfying case I.

Case II: $D_1+D_2+D_3+D_4=9$. Here we have $\{D_1,D_2,D_3,D_4\}=\{0,2,3,4\}$, $D_1\neq 0$ or $\{D_1,D_2,D_3,D_4\}=\{0,1,3,5\}$, $D_1\neq 0$. Like before, there are $3\cdot 3!=18$ numbers in each possibility, thus we have $2\times 18=36$ numbers in case II.

Case III: $D_1+D_2+D_3+D_4=12$. Here we have $\{D_1,D_2,D_3,D_4\}=\{0,3,4,5\}$, $D_1\neq 0$ or $\{D_1,D_2,D_3,D_4\}=\{1,2,4,5\}$. In the first possibility there are $3\cdot 3!=18$ numbers, and in the second there are $4!=24$. Thus we have $18+24=42$ numbers in case III.

The desired number is thus $18+36+42=96$.

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7 Combinatorics: The Inclusion-Exclusion Principle

In this section we investigate a tool for counting unions of events. It is known as *The Principle of Inclusion-Exclusion* or Sylvester-Poincaré Principle. We use $\text{card}\,(X)$ or $|X|$ to denote the cardinality, or size, of the set X. We will use that fact that if X and Y are *disjoint* sets, then $\text{card}\,(X \cup Y) = \text{card}\,(X) + \text{card}\,(Y)$.

Theorem 7.1 Two set Inclusion-Exclusion.

$$\text{card}\,(A \cup B) = \text{card}\,(A) + \text{card}\,(B) - \text{card}\,(A \cap B)$$

Proof 1. We have

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B),$$

and this last expression is a union of disjoint sets. Hence

$$\text{card}\,(A \cup B) = \text{card}\,(A \setminus B) + \text{card}\,(B \setminus A) + \text{card}\,(A \cap B)\,.$$

But

$$A \setminus B = A \setminus (A \cap B) \implies \text{card}\,(A \setminus B) = \text{card}\,(A) - \text{card}\,(A \cap B)\,,$$

$$B \setminus A = B \setminus (A \cap B) \implies \text{card}\,(B \setminus A) = \text{card}\,(B) - \text{card}\,(A \cap B)\,,$$

which proves the result.

Proof 2. Construct a Venn diagram, and mark by R_1 the number of elements which are simultaneously in both sets (i.e., in $A \cap B$), by R_2 the number of elements which are in A but not in B (i.e., in $A \setminus B$), and by R_3 the number of elements which are B but not in A (i.e., in $B \setminus A$). We have $R_2 + R_3 - R_1 = \text{card}\,(A \cup B)$, which illustrates the theorem.

Example 7.1 Of 40 people, 28 swim and 16 run. It is also known that 10 both swim and run. How many among the 40 neither swim nor run?

Solution: Let A denote the set of swimmers and B the set of runners. Then

$$\text{card}\,(A \cup B) = \text{card}\,(A) + \text{card}\,(B) - \text{card}\,(A \cap B) = 28 + 16 - 10 = 34,$$

meaning that there are 34 people that either swim or run (or possibly both). Therefore the number of people that neither swim nor run is $40 - 34 = 6$.

Example 7.2 How many integers between 1 and 1000, inclusive, do not share a common factor with 1000?

Solution: Observe that $1000 = 2^3 5^3$, and thus from the 1000 integers we must weed out those that have a factor of 2 or of 5 in their prime factorization. If A_2 denotes the set of those integers divisible by 2 in the interval $[1, 1000]$ then clearly $\text{card}(A_2) = \lfloor \frac{1000}{2} \rfloor = 500$. Similarly, if A_5 denotes the set of those integers divisible by 5 then $\text{card}(A_5) = \lfloor \frac{1000}{5} \rfloor = 200$. Also $\text{card}(A_2 \cap A_5) = \lfloor \frac{1000}{10} \rfloor = 100$. This means that there are $\text{card}(A_2 \cup A_5) = 500 + 200 - 100 = 600$ integers in the interval $[1, 1000]$ sharing at least a factor with 1000, thus there are $1000 - 600 = 400$ integers in $[1, 1000]$ that do not share a factor prime factor with 1000.

We now deduce a formula for counting the number of elements of a union of three events.

Theorem 7.2 Three set Inclusion-Exclusion. Let A, B, C be events of the same sample space Ω. Then

$$\begin{aligned} \text{card}(A \cup B \cup C) &= \text{card}(A) + \text{card}(B) + \text{card}(C) \\ &\quad - \text{card}(A \cap B) - \text{card}(B \cap C) - \text{card}(C \cap A) \\ &\quad + \text{card}(A \cap B \cap C) \end{aligned}$$

Proof 1. Using the associativity and distributivity of unions of sets, we see that

$$\begin{aligned} \text{card}(A \cup B \cup C) &= \text{card}(A \cup (B \cup C)) \\ &= \text{card}(A) + \text{card}(B \cup C) - \text{card}(A \cap (B \cup C)) \\ &= \text{card}(A) + \text{card}(B \cup C) - \text{card}((A \cap B) \cup (A \cap C)) \\ &= \text{card}(A) + \text{card}(B) + \text{card}(C) - \text{card}(B \cap C) \\ &\quad - \text{card}(A \cap B) - \text{card}(A \cap C) \\ &\quad + \text{card}((A \cap B) \cap (A \cap C)) \\ &= \text{card}(A) + \text{card}(B) + \text{card}(C) - \text{card}(B \cap C) \\ &\quad - (\text{card}(A \cap B) + \text{card}(A \cap C) - \text{card}(A \cap B \cap C)) \\ &= \text{card}(A) + \text{card}(B) + \text{card}(C) \\ &\qquad - \text{card}(A \cap B) - \text{card}(B \cap C) - \text{card}(C \cap A) \\ &\qquad + \text{card}(A \cap B \cap C). \end{aligned}$$

This gives the Inclusion-Exclusion Formula for three sets.

Proof 2. Construct a Venn Diagram.

Example 7.3 How many integers between 1 and 600 inclusive are not divisible by 3, 5, or 7?

Solution: Let A_k denote the set of integers between 1 and 600 which are divisible by k. Then

$$\begin{aligned}
\text{card}\,(A_3) &= \lfloor \frac{600}{3} \rfloor = 200, \\
\text{card}\,(A_5) &= \lfloor \frac{600}{5} \rfloor = 120, \\
\text{card}\,(A_7) &= \lfloor \frac{600}{7} \rfloor = 85, \\
\text{card}\,(A_{15}) &= \lfloor \frac{600}{15} \rfloor = 40 \\
\text{card}\,(A_{21}) &= \lfloor \frac{600}{21} \rfloor = 28 \\
\text{card}\,(A_{35}) &= \lfloor \frac{600}{35} \rfloor = 17 \\
\text{card}\,(A_{105}) &= \lfloor \frac{600}{105} \rfloor = 5
\end{aligned}$$

By Inclusion-Exclusion there are $200 + 120 + 85 - 40 - 28 - 17 + 5 = 325$ integers between 1 and 600 divisible by at least one of 3, 5, or 7. Those not divisible by these numbers are a total of $600 - 325 = 275$.

Example 7.4 In a group of 30 people, 8 speak English, 12 speak Spanish and 10 speak French. It is known that 5 speak English and Spanish, 5 Spanish and French, and 7 English and French. The number of people speaking all three languages is 3. How many do not speak any of these languages?

Solution: Let A be the set of all English speakers, B the set of Spanish speakers and C the set of French speakers in our group. Construct a Venn diagram. In the intersection of all three we put 8. In the region common to A and B which is not filled up we put $5 - 2 = 3$. In the region common to A and C which is not already filled up we put $5 - 3 = 2$. In the region common to B and C which is not already filled up, we put $7 - 3 = 4$. In the remaining part of A we put $8 - 2 - 3 - 2 = 1$, in the remaining part of B we put $12 - 4 - 3 - 2 = 3$, and in the remaining part of C we put $10 - 2 - 3 - 4 = 1$. Each of the mutually disjoint regions comprise a total of $1 + 2 + 3 + 4 + 1 + 2 + 3 = 16$ persons. Those outside these three sets are then $30 - 16 = 14$.

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8 Combinatorics: Permutations and Combinations

8.1 Permutations without Repetitions

Definition 8.1 We define the symbol ! (factorial), as follows: $0! = 1$, and for integers $n \geq 1$,

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

$n!$ is read n *factorial*.

Example 8.1 We have

$$\begin{aligned} 1! &= 1, \\ 2! &= 1 \cdot 2 = 2, \\ 3! &= 1 \cdot 2 \cdot 3 = 6, \\ 4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24, \\ 5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120. \end{aligned}$$

Example 8.2 We have

$$\begin{aligned} \frac{7!}{4!} &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 210, \\ \frac{(n+2)!}{n!} &= \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1), \\ \frac{(n-2)!}{(n+1)!} &= \frac{(n-2)!}{(n+1)(n)(n-1)(n-2)!} = \frac{1}{(n+1)(n)(n-1)}. \end{aligned}$$

Definition 8.2 Let $x_1, x_2, \ldots, x_n$ be n distinct objects. A **permutation** of these objects is simply a rearrangement of them.

Example 8.3 There are 24 permutations of the letters in $MATH$, namely

$$\begin{array}{cccccc} MATH & MAHT & MTAH & MTHA & MHTA & MHAT \\ AMTH & AMHT & ATMH & ATHM & AHTM & AHMT \\ TAMH & TAHM & TMAH & TMHA & THMA & THAM \\ HATM & HAMT & HTAM & HTMA & HMTA & HMAT \end{array}$$

Theorem 8.1 Permutations. Let $x_1, x_2, \ldots, x_n$ be n distinct objects. Then there are $n!$ permutations of them.

Proof. The first position can be chosen in n ways, the second object in $n-1$ ways, the third in $n-2$, etc. This gives

$$n(n-1)(n-2)\cdots 2 \cdot 1 = n!.$$

Example 8.4 The number of permutations of the letters of the word *RETICULA* is $8! = 40320$.

Example 8.5 A bookshelf contains 5 German books, 7 Spanish books and 8 French books. Each book is different from one another.

(a) How many different arrangements can be made of these books?

(b) How many different arrangements can be made of these books if books of each language must be next to each other?

(c) How many different arrangements can be made of these books if all the French books must be next to each other?

(d) How many different arrangements can be made of these books if no two French books can be next to each other?

Solution:

(a) We are permuting $5 + 7 + 8 = 20$ objects. Thus the number of arrangements sought is $20! = 2432902008176640000$.

(b) "Glue" the books by language, this will assure that books of the same language are together. We permute the 3 languages in 3! ways. We permute the German books in 5! ways, the Spanish books in 7! ways and the French books in 8! ways. Hence the total number of ways is $3!5!7!8! = 146313216000$.

(c) Align the German books and the Spanish books first. Putting these $5 + 7 = 12$ books creates $12 + 1 = 13$ spaces (we count the space before the first book, the spaces between books and the space after the last book). To assure that all the French books are next each other, we "glue" them together and put them in one of these spaces. Now, the French books can be permuted in 8! ways and the non-French books can be permuted in 12! ways. Thus the total number of permutations is
$$(13)8!12! = 251073478656000.$$

(d) Align the German books and the Spanish books first. Putting these $5 + 7 = 12$ books creates $12 + 1 = 13$ spaces (we count the space before the first book, the spaces between books and the space after the last book). To assure that no two French books are next to each other, we put them into these spaces. The first French book can be put into any of 13 spaces, the second into any of 12, etc., the eighth French book can be put into any 6 spaces. Now, the non-French books can be permuted in 12! ways. Thus the total number of permutations is
$$(13)(12)(11)(10)(9)(8)(7)(6)12!,$$
which is 24856274386944000.

8.2 Permutations with Repetitions

We now consider permutations with repeated objects.

Example 8.6 In how many ways may the letters of the word

$$MASSACHUSETTS$$

be permuted?

Solution: We put subscripts on the repeats forming

$$MA_1S_1S_2A_2CHUS_3ET_1T_2S_4.$$

There are now 13 distinguishable objects, which can be permuted in 13! different ways. For each of these 13! permutations, A_1A_2 can be permuted in 2! ways, $S_1S_2S_3S_4$ can be permuted in 4! ways, and T_1T_2 can be permuted in 2! ways. Thus the over count 13! is corrected by the total actual count

$$\frac{13!}{2!4!2!} = 64864800.$$

In general, we may prove the following theorem.

Theorem 8.2 Permutations with Repetitions. Let there be k types of objects: n_1 of type 1; n_2 of type 2; etc. Then the number of ways in which these $n_1 + n_2 + \cdots + n_k$ objects can be rearranged is

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!n_2!\cdots n_k!}.$$

Example 8.7 In how many ways may we permute the letters of the word $MASSACHUSETTS$ in such a way that $MASS$ is always together, in this order?

Solution: The particle $MASS$ can be considered as one block and the 9 letters A, C, H, U, S, E, T, T, S. In A, C, H, U, S, E, T, T, S there are four S's and two T's and so the total number of permutations sought is

$$\frac{10!}{2!2!} = 907200.$$

Example 8.8 In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, $1 + 7 + 1$ is to be regarded different from $7 + 1 + 1$.

Solution: We first look for answers with

$$a + b + c = 9, \quad 1 \le a \le b \le c \le 7$$

and we find the permutations of each triplet. We have

(a, b, c)	Number of permutations
$(1, 1, 7)$	$\frac{3!}{2!} = 3$
$(1, 2, 6)$	$3! = 6$
$(1, 3, 5)$	$3! = 6$
$(1, 4, 4)$	$\frac{3!}{2!} = 3$
$(2, 2, 5)$	$\frac{3!}{2!} = 3$
$(2, 3, 4)$	$3! = 6$
$(3, 3, 3)$	$\frac{3!}{3!} = 1$

Thus the number desired is

$$3 + 6 + 6 + 3 + 3 + 6 + 1 = 28.$$

Example 8.9 In how many ways can the letters of the word **MURMUR** be arranged without letting two letters which are alike come together?

Solution: If we started with, say , **MU** then the **R** could be arranged as follows:

M	**U**	**R**		**R**	

,

M	**U**	**R**			**R**

,

M	**U**		**R**		**R**

.

In the first case there are $2! = 2$ ways of putting the remaining **M** and **U**. In the second case, there are $2! = 2$ ways, and in the third case there is only $1! = 1$ way. Thus, starting the word with **MU** gives $2 + 2 + 1 = 5$ possible arrangements. In the general case, we can choose the first letter of the word in 3 ways, and the second in 2 ways. Thus the number of arrangements sought is $3 \cdot 2 \cdot 5 = 30$.

8.3 Combinations without Repetitions

Definition 8.3 Let n, k be non-negative integers with $0 \leq k \leq n$. The symbol $\binom{n}{k}$ (read "*n choose k*") is defined and denoted by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{1 \cdot 2 \cdot 3 \cdots k}.$$

Remark. Observe that in the last fraction, there are k factors in both the numerator and denominator. Also, observe the boundary conditions

$$\binom{n}{0} = \binom{n}{n} = 1, \qquad \binom{n}{1} = \binom{n}{n-1} = n.$$

Example 8.10 We have

$$\begin{aligned}\binom{6}{3} &= \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3} = 20,\\ \binom{11}{2} &= \frac{11\cdot 10}{1\cdot 2} = 55,\\ \binom{12}{7} &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = 792,\\ \binom{110}{109} &= 110,\\ \binom{110}{0} &= 1,\\ \binom{110}{1} &= 110.\end{aligned}$$

Remark. Since $n-(n-k)=k$, we have, for any integers n,k such that $0\le k\le n$, the symmetry identity

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}.$$

This can be interpreted as follows: if there are n different tickets in a hat, choosing k of them out of the hat is the same as choosing $n-k$ of them to remain in the hat.

Example 8.11

$$\binom{11}{9} = \binom{11}{2} = 55,$$
$$\binom{12}{5} = \binom{12}{7} = 792.$$

Definition 8.4 Suppose that we have n distinguishable objects. A *k-combination* is a selection of k, $(0\le k\le n)$ objects from the n made without regards to order.

Example 8.12 The 2-combinations from the list $\{X,Y,Z,W\}$ are

$$XY, XZ, XW, YZ, YW, WZ.$$

Example 8.13 The 3-combinations from the list $\{X,Y,Z,W\}$ are

$$XYZ, XYW, XZW, YWZ.$$

Theorem 8.3 Combinations. Let there be n distinguishable objects, and let k be an integer such that $0\le k\le n$. Then the numbers of k-combinations of these n

objects is $\binom{n}{k}$.

Proof. Pick any of the k objects. This can be done in $n(n-1)(n-2)\cdots(n-k+1)$ ways, since there are n ways of choosing the *first*, $n-1$ ways of choosing the *second*, etc. This particular choice of k objects can be permuted in $k!$ ways. Hence the total number of k-combinations is

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \binom{n}{k}.$$

Example 8.14 From a group of 10 people, we can choose a committee of 4 in $\binom{10}{4} = 210$ ways.

Example 8.15 In a group of 2 camels, 3 goats, and 10 sheep in how many ways can we choose a committee of 6 animals if

(a) there are no constraints in species?

(b) the two camels must be included?

(c) the two camels must be excluded?

(d) there must be at least 3 sheep?

(e) there must be at most 2 sheep?

(f) Joe Camel, Billy Goat and Samuel Sheep hate each other and they will not work in the same group. How many compatible committees are there?

Solution:

(a) There are $2+3+10=15$ animals and we must choose 6. Thus, there are

$$\binom{15}{6} = 5005$$

possible committees.

(b) Since the 2 camels are included, we must choose $6-2=4$ more animals from a list of $15-2=13$ animals, so there are

$$\binom{13}{4} = 715$$

possible committees.

(c) Since the 2 camels must be excluded, we must choose 6 animals from a list of $15 - 2 = 13$, so there are

$$\binom{13}{6} = 1716$$

possible committees.

(d) If k sheep are chosen from the 10 sheep, $6 - k$ animals must be chosen from the remaining 5 animals, so there are

$$\binom{10}{3}\binom{5}{3} + \binom{10}{4}\binom{5}{2} + \binom{10}{5}\binom{5}{1} + \binom{10}{6}\binom{5}{0} = 4770$$

possible committees.

(e) First observe that there cannot be 0 sheep, since that would mean choosing 6 other animals. Hence, there must be either 1 or 2 sheep, and so 3 or 4 of the other animals. The total number is thus

$$\binom{10}{2}\binom{5}{4} + \binom{10}{1}\binom{5}{5} = 235.$$

(f) A compatible group will either exclude all these three animals or include exactly one of them. This can be done in

$$\binom{12}{6} + \binom{3}{1}\binom{12}{5} = 3300$$

ways.

Example 8.16 Consider the set of 5-digit positive integers written in decimal notation.

(a) How many are there?

(b) How many do not have a 9 in their decimal representation?

(c) How many have at least one 9 in their decimal representation?

(d) How many have exactly one 9?

(e) How many have exactly two 9's?

(f) How many have exactly three 9's?

(g) How many have exactly four 9's?

(h) How many have exactly five 9's?

(i) How many have neither an 8 nor a 9 in their decimal representation?

(j) How many have neither a 7, nor an 8, nor a 9 in their decimal representation?

(k) How many have either a 7, an 8, or a 9 in their decimal representation?

Solution:

(a) There are 9 possible choices for the first digit and 10 possible choices for the remaining digits. The number of choices is thus $9 \cdot 10^4 = 90000$.

(b) There are 8 possible choices for the first digit and 9 possible choices for the remaining digits. The number of choices is thus $8 \cdot 9^4 = 52488$.

(c) The difference $90000 - 52488 = 37512$.

(d) We condition on the first digit. If the first digit is a 9 then the other four remaining digits must be different from 9, giving $9^4 = 6561$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{1} = 4$ ways of choosing were the 9 will be, and we have 9^3 ways of filling the 3 remaining spots. Thus in this case there are $8 \cdot 4 \cdot 9^3 = 23328$ such numbers. In total there are $6561 + 23328 = 29889$ five-digit positive integers with exactly one 9 in their decimal representation.

(e) We condition on the first digit. If the first digit is a 9 then one of the remaining four must be a 9, and the choice of place can be accomplished in $\binom{4}{1} = 4$ ways. The other three remaining digits must be different from 9, giving $4 \cdot 9^3 = 2916$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{2} = 6$ ways of choosing were the two 9's will be, and we have 9^2 ways of filling the two remaining spots. Thus in this case there are $8 \cdot 6 \cdot 9^2 = 3888$ such numbers. Altogether there are $2916 + 3888 = 6804$ five-digit positive integers with exactly two 9's in their decimal representation.

(f) Again we condition on the first digit. If the first digit is a 9 then two of the remaining four must be 9's, and the choice of place can be accomplished in $\binom{4}{2} = 6$ ways. The other two remaining digits must be different from 9, giving $6 \cdot 9^2 = 486$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{3} = 4$ ways of choosing were the three 9's will be, and we have 9 ways of filling the remaining spot. Thus in this case there are $8 \cdot 4 \cdot 9 = 288$ such numbers. Altogether there are $486 + 288 = 774$ five-digit positive integers with exactly three 9's in their decimal representation.

(g) If the first digit is a 9 then three of the remaining four must be 9's, and the choice of place can be accomplished in $\binom{4}{3} = 4$ ways. The other remaining digit must be different from 9, giving $4 \cdot 9 = 36$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{4} = 4$ ways of choosing were the four 9's will be, thus filling all the spots. Thus in this case there are $8 \cdot 1 = 8$ such numbers. Altogether there are $36 + 8 = 44$ five-digit positive integers with exactly three 9's in their decimal representation.

(h) There is obviously only 1 such positive integer.

Remark. Observe that $37512 = 29889 + 6804 + 774 + 44 + 1$.

(i) We have 7 choices for the first digit and 8 choices for the remaining 4 digits, giving $7 \cdot 8^4 = 28672$ such integers.

(j) We have 6 choices for the first digit and 7 choices for the remaining 4 digits, giving $6 \cdot 7^4 = 14406$ such integers.

(k) We use inclusion-exclusion. The desired number is $90000 - 85854 = 4146$.

8.4 Combinations with Repetitions

Theorem 8.4 Theorem (DeMoivre). Let n be a positive integer. The number of positive integer solutions to

$$x_1 + x_2 + \cdots + x_r = n$$

is

$$\binom{n-1}{r-1}.$$

Proof. Write n as

$$n = 1 + 1 + \cdots + 1 + 1,$$

where there are n 1's and $n-1$ plus signs. To decompose n in r summands we need to choose $r-1$ plus signs from the $n-1$, which proves the theorem.

Example 8.17 In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, $1+7+1$ is to be regarded different from $7+1+1$.

Solution: We are seeking integral solutions to

$$a + b + c = 9, \quad a > 0, b > 0, c > 0.$$

The number of solutions is thus

$$\binom{9-1}{3-1} = \binom{8}{2} = 28.$$

Example 8.18 In how many ways can 100 be written as the sum of four positive integer summands?

Solution: We want the number of positive integer solutions to

$$a + b + c + d = 100,$$

which is

$$\binom{99}{3} = 156849.$$

Corollary 8.5 Let n be a positive integer. The number of non-negative integer solutions to

$$y_1 + y_2 + \cdots + y_r = n$$

is

$$\binom{n+r-1}{r-1}.$$

Proof. Set $x_r - 1 = y_r$. Then $x_r \geq 1$. The equation

$$x_1 - 1 + x_2 - 1 + \cdots + x_r - 1 = n$$

is equivalent to

$$x_1 + x_2 + \cdots + x_r = n + r,$$

which has

$$\binom{n+r-1}{r-1}$$

solutions.

Example 8.19 Find the number of quadruples (a, b, c, d) of integers satisfying

$$a + b + c + d = 100, \; a \geq 30, b > 21, c \geq 1, d \geq 1.$$

Solution: Set $a' + 29 = a, b' + 20 = b$. Then we want the number of positive integer solutions to

$$a' + 29 + b' + 21 + c + d = 100,$$

or

$$a' + b' + c + d = 50.$$

This number is

$$\binom{49}{3} = 18424.$$

Example 8.20 There are five people in a lift of a building having eight floors. In how many ways can they choose their floor for exiting the lift?

Solution: Let x_i be the number of people that floor i receives. We are looking for non-negative solutions of the equation

$$x_1 + x_2 + \cdots + x_8 = 5.$$

Setting $y_i = x_i + 1$, then

$$\begin{aligned} x_1 + x_2 + \cdots + x_8 = 5 \quad &\Longrightarrow \quad (y_1 - 1) + (y_2 - 1) + \cdots + (y_8 - 1) = 5 \\ &\Longrightarrow \quad y_1 + y_2 + \cdots + y_8 = 13, \end{aligned}$$

so the number sought is the number of positive solutions to

$$y_1 + y_2 + \cdots + y_8 = 13,$$

which is $\binom{12}{7} = 792$.

Example 8.21 Find the number of quadruples (a, b, c, d) of non-negative integers which satisfy the inequality

$$a + b + c + d \leq 2001.$$

Solution: The number of non-negative solutions to

$$a + b + c + d \leq 2001$$

is equal to the number of solutions to

$$a + b + c + d + f = 2001$$

where f is a non-negative integer. This number is the same as the number of positive integer solutions to

$$a_1 - 1 + b_1 - 1 + c_1 - 1 + d_1 - 1 + f_1 - 1 = 2001,$$

which is $\binom{2005}{4}$.

Example 8.22 How many integral solutions to the equation

$$a + b + c + d = 100,$$

are there given the following constraints:

$$1 \leq a \leq 10,\ \ b \geq 0,\ \ c \geq 2, 20 \leq d \leq 30?$$

Solution: We use Inclusion-Exclusion. There are $\binom{80}{3} = 82160$ integral solutions to

$$a + b + c + d = 100, \quad a \geq 1, b \geq 0, c \geq 2, d \geq 20.$$

Let A be the set of solutions with

$$a \geq 11, b \geq 0, c \geq 2, d \geq 20$$

and B be the set of solutions with

$$a \geq 1, b \geq 0, c \geq 2, d \geq 31.$$

Then $\operatorname{card}(A) = \binom{70}{3}$, $\operatorname{card}(B) = \binom{69}{3}$, $\operatorname{card}(A \cap B) = \binom{59}{3}$ and so

$$\operatorname{card}(A \cup B) = \binom{70}{3} + \binom{69}{3} - \binom{59}{3} = 74625.$$

The total number of solutions to

$$a + b + c + d = 100$$

with

$$1 \leq a \leq 10,\ \ b \geq 0,\ \ c \geq 2, 20 \leq d \leq 30$$

is thus

$$\binom{80}{3} - \binom{70}{3} - \binom{69}{3} + \binom{59}{3} = 7535.$$

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9 Combinatorics: The Binomial Theorem and Some Combinatorial Identities

We recall that the symbol

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

counts the number of ways of selecting k different objects from n different objects.

Example 9.1 Prove *Newton's Identity*:

$$\binom{n}{i}\binom{i}{j} = \binom{n}{j}\binom{n-j}{j-i},$$

for integers $0 \leq j \leq i \leq n$.

Solution #1: We have

$$\binom{n}{i}\binom{i}{j} = \frac{n!i!}{i!(n-i)!j!(i-j)!} = \frac{n!(n-j)!}{(n-j)!j!(n-i)!(i-j)!},$$

which is equal to

$$\binom{n}{j}\binom{n-j}{i-j}.$$

Solution #2: Let's think about this identity from a combinatorial point of view. Consider a group of n people from which we want to form an i-member committee with j leaders. We can choose i people to be on a committee in $\binom{n}{i}$ ways, and then choose the j committee leaders in $\binom{i}{j}$ ways. Or, we can choose the j leaders first in $\binom{n}{j}$ ways, and then choose the remaining $j-i$ committee members from the remaining $n-j$ people in $\binom{n-j}{j-i}$ ways. Since both methods will give us all of the committees, we have

$$\binom{n}{i}\binom{i}{j} = \binom{n}{j}\binom{n-j}{j-i}.$$

Example 9.2 Prove *Pascal's Identity*:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

for integers $1 \leq k \leq n$.

Solution #1: We have

$$\begin{aligned}
\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{(n-1)!}{(n-k-1)!(k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right) \\
&= \frac{(n-1)!}{(n-k-1)!(k-1)!}\frac{n}{(n-k)k} \\
&= \frac{n!}{(n-k)!k!}. \\
&= \binom{n}{k}
\end{aligned}$$

Solution #2: Let's think about this identity from a combinatorial point of view. We know that the left-hand side, $\binom{n}{k}$ is the number of ways that we can choose k objects from a set of n objects. Now, suppose we mark a particular object "X" in the set of n objects. Then, every time we choose k objects from the set of n objects, either "X" is in the set, or it is not. If object "X" is in the set, then we only need to choose $k-1$ objects from the remaining $n-1$ objects (since we already know that "X" has been chosen). This can be done in $\binom{n-1}{k-1}$ ways. If "X" is not in the set, then we need to choose k objects from the remaining $n-1$ objects. This can be done in $\binom{n-1}{k}$ ways. Thus, we conclude that

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}.$$

Pascal's Triangle. Using Pascal's Identity, we obtain *Pascal's Triangle.* Each number is equal to the sum of the two numbers above it (to the left and right).

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

Substituting the numerical values of the binomial coefficients, we have:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

We see from Pascal's Triangle that the binomial coefficients are symmetric. This symmetry is easily justified algebraically by the identity $\binom{n}{k} = \binom{n}{n-k}$. We also notice that the binomial coefficients tend to increase until they reach the middle, and that they decrease symmetrically. We call this property the *unimodality* of the binomial coefficients. For example, without finding the exact numerical values we can see that $\binom{200}{17} < \binom{200}{69}$ and that $\binom{200}{131} = \binom{200}{69} < \binom{200}{99}$.

Next, we use Pascal's Triangle in order to expand the binomial

$$(a+b)^n.$$

Definition 9.1 Sigma Notation. The expression

$$\sum_{k=0}^{n} a_k$$

is the sum of all of the terms $a_0,\ a_1, \ldots, a_n$:

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \cdots a_n.$$

Example 9.3 $\displaystyle\sum_{k=0}^{5} 3^k = 3^0 + 3^1 + 3^2 + 3^3 + 3^4 + 3^5.$

Example 9.4 $\displaystyle\sum_{j=1}^{4} j^2 = 1^2 + 2^2 + 3^2 + 4^2.$

Theorem 9.1 Binomial Theorem, Part 1. For any n such that $n \geq 0$, we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Proof. Observe that expanding

$$\underbrace{(1+x)(1+x)\cdots(1+x)}_{n \text{ factors}}$$

consists of adding up all the terms obtained from multiplying either a 1 or a x from the first set of parentheses times either a 1 or an x from the second set of parentheses etc. To get x^k, x must be chosen from exactly k of the sets of parentheses. Thus the number of x^k terms is $\binom{n}{k}$. It follows that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$

Example 9.5

$$\begin{aligned}
(1+x)^2 &= \sum_{k=0}^{2}\binom{2}{k}x^k \\
&= \binom{2}{0}x^0 + \binom{2}{1}x^1 + \binom{2}{2}x^2 \\
&= 1 + 2x + x^2 \\
(1+x)^3 &= \sum_{k=0}^{3}\binom{3}{k}x^k \\
&= \binom{3}{0}x^0 + \binom{3}{1}x^1 + \binom{3}{2}x^2 + \binom{3}{3}x^3 \\
&= 1 + 3x + 3x^2 + x^3 \\
(1+x)^4 &= \sum_{k=0}^{4}\binom{4}{k}x^k \\
&= \binom{4}{0}x^0 + \binom{4}{1}x^1 + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{3}x^4 \\
&= 1 + 4x + 6x^2 + 4x^3 + x^4
\end{aligned}$$

Theorem 9.2 Binomial Theorem, Part 2: More generally, for any n such that $n \geq 0$ and for any a, b, we have

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}.$$

Example 9.6

$$\begin{aligned}(a+b)^2 &= \sum_{k=0}^{2}\binom{2}{k}a^k b^{2-k}\\ &= \binom{2}{0}a^0b^2+\binom{2}{1}a^1b^1+\binom{2}{2}a^2b^0\\ &= b^2+2ab+a^2\\ (a+b)^3 &= \sum_{k=0}^{3}\binom{3}{k}a^k b^{3-k}\\ &= \binom{3}{0}a^0b^3+\binom{3}{1}a^1b^2+\binom{3}{2}a^2b^1+\binom{3}{3}a^3b^0\\ &= b^3+3ab^2+3a^2b+a^3\end{aligned}$$

Example 9.7 Prove that

$$\sum_{k=0}^{n}\binom{n}{k}=2^n.$$

Solution: This follows from letting $x=1$ in the expansion

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k.$$

Can you give a combinatorial proof of this result?

Example 9.8 Prove that for integer $n \geq 1$,

$$\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}=\binom{n}{i}2^{n-i},\ i\leq n.$$

Solution: Recall that by Newton's Identity

$$\binom{n}{j}\binom{j}{i}=\binom{n}{i}\binom{n-i}{j-i}.$$

Thus

$$\sum_{j=0}^{n}\binom{n}{j}\binom{j}{i}=\binom{n}{i}\sum_{j=0}^{n}\binom{n-i}{j-i}.$$

Re-indexing the sum on the right, we obtain:

$$\sum_{j=0}^{n}\binom{n-i}{j-i}=\sum_{j=0}^{n-i}\binom{n-i}{j}=2^{n-i},$$

by the preceding problem. Thus the assertion follows.

Example 9.9 Simplify

$$\sum_{0\leq k\leq 50}\binom{100}{2k}.$$

Solution: By the Binomial Theorem

$$\begin{aligned}(1+1)^{100} &= \binom{100}{0}+\binom{100}{1}+\binom{100}{2}+\ldots+\binom{100}{99}+\binom{100}{100} \\ (1-1)^{100} &= \binom{100}{0}-\binom{100}{1}+\binom{100}{2}-\ldots-\binom{100}{99}+\binom{100}{100},\end{aligned}.$$

Adding these two equations together, we obtain:

$$2^{100}=2\binom{100}{0}+2\binom{100}{2}+\ldots+2\binom{100}{100}.$$

Dividing by 2, the required sum is thus 2^{99}.

Example 9.10 Simplify

$$\sum_{k=1}^{10}2^k\binom{11}{k}.$$

Solution: By the Binomial Theorem, the complete sum $\sum_{k=0}^{11}\binom{11}{k}2^k = 3^{11}$. The required sum lacks the zeroth term, $\binom{11}{0}2^0=1$, and the eleventh term, $\binom{11}{11}2^{11}$ from this complete sum. The required sum is thus $3^{11}-2^{11}-1$.

Example 9.11 Which coefficient of the expansion of

$$\left(\frac{1}{3}+\frac{2}{3}x\right)^{10}$$

has the greatest magnitude?

Solution: By the Binomial Theorem,

$$\left(\frac{1}{3}+\frac{2}{3}x\right)^{10}=\sum_{k=0}^{10}\binom{10}{k}(1/3)^k(2x/3)^{10-k}=\sum_{k=0}^{10}a_kx^k.$$

We consider the ratios $\frac{a_k}{a_{k-1}}, k=1,2,\ldots n$. This ratio is seen to be

$$\frac{a_k}{a_{k-1}}=\frac{2(10-k+1)}{k}.$$

This will be < 1 if $k < 22/3 < 8$. Thus $a_0 < a_1 < a_2 < \ldots < a_7$. If $k > 22/3$, the ratio above will be < 1. Thus $a_7 > a_8 > a_9 > a_{10}$. The largest term is that of $k = 7$, i.e. the eighth term.

Example 9.12 At what positive integral value of x is the x^4 term in the expansion of $(2x+9)^{10}$ greater than the adjacent terms?

Solution: We want to find integral x such that

$$\binom{10}{4}(2x)^4(9)^6 \geq \binom{10}{3}(2x)^3(9)^7,$$

and

$$\binom{10}{4}(2x)^4(9)^6 \geq \binom{10}{5}(2x)^5(9)^5.$$

After simplifying the factorials, the two inequalities sought are

$$x \geq 18/7$$

and

$$15/4 \geq x.$$

The only integral x that satisfies this is $x = 3$.

Example 9.13 Prove that if m, n are nonnegative integers then

$$\binom{n+1}{m+1} = \sum_{k=m}^{n} \binom{k}{m}.$$

Solution: Using Pascal's Identity, we obtain:

$$\begin{aligned}
\sum_{k=m}^{n} \binom{k}{m} &= \binom{m}{m+1} + \binom{m}{m} + \binom{m+1}{m} + \cdots + \binom{n}{m} \\
&= \binom{m+1}{m+1} + \binom{m+1}{m} + \binom{m+2}{m} + \cdots + \binom{n}{m} \\
&= \binom{m+2}{m+1} + \binom{m+2}{m} + \binom{m+3}{m} + \cdots + \binom{n}{m} \\
&\vdots \\
&= \binom{n}{m+1} + \binom{n}{m} \\
&= \binom{n+1}{m+1}.
\end{aligned}$$

10 Probability

Definition 10.1 The **probability** that an event occurs is defined as

$$P = \frac{\text{The number of distinct ways that the event can occur}}{\text{The total number of all possible outcomes}}.$$

Example 10.1 The probability of randomly drawing a face card from a deck of 52 ordinary playing cards is $\frac{12}{52}$.

Example 10.2 Find the probability that when two cards are randomly drawn, without replacing the first card, at least one of the cards is a face card.

Solution: We must consider the following three cases separately:

- Case 1: The first card is a face card and the second card is not a face card. This can happen in $12 \cdot 40$ different ways.
- Case 2: The first card is a not a face card and the second card is a face card. This can happen in $40 \cdot 12$ different ways.
- Case 3: Both cards are face cards. This can happen in $12 \cdot 11$ different ways.

Thus, the total number of ways in which at least one of the cards can be a face card is $12 \cdot 40 + 40 \cdot 12 + 12 \cdot 11 = 1092$. The total number of ways to draw two cards (without replacing the first) is $52 \cdot 51$. Thus, the probability that at least one of the cards is a face card is

$$P = \frac{1092}{52 \cdot 51} = \frac{7}{17}.$$

Theorem 10.1 We can obtain **some basic results** about probabilities directly from the definition. Let $P(A)$ denote the probability that event A occurs.

- $0 \leq P(A) \leq 1$.
- The probability that A does not occur is $1 - P(A)$.
- $P(A \cup B)$ is the probability that event A OR event B occurs. $P(A \cap B)$ is the probability that event A AND event B both occur. If A and B are two events, then
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
- If A and B are two *independent* events (i.e. events which do not affect each other's outcomes), then
$$P(A \cap B) = P(A) \cdot P(B).$$

Example 10.3 Find the probability that a king or a heart is drawn when choosing one card from an ordinary deck of 52 cards.

Solution:

$$\begin{aligned} P(\text{heart or king}) &= P(\text{heart}) + P(\text{king}) - P(\text{heart and king}) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13}. \end{aligned}$$

Example 10.4 (2003 AMC 10B #21) A bag contains two red beads and two green beads. Reach into the bag and pull out a bead, replacing it with a red bead regardless of the color. What is the probability that all the beads are red after three such replacements?

(A) 1/8 (B) 5/32 (C) 9/32 (D) 3/8 (E) 7/16

Solution: For all the beads to be red after three replacements, two of the three beads chosen must have been green. Thus, the selections must have been one of the following:

- Case 1: red, green, green. The probability of this case is
$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{16}.$$
- Case 2: green, red, green. The probability of this case is
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{32}.$$
- Case 3: green, green, red. The probability of this case is
$$\frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{8}.$$

Thus, the desired probability is

$$\frac{1}{16} + \frac{3}{32} + \frac{1}{8} = \frac{9}{32}.$$

Example 10.5 (2004 AMC 10A #10) Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

(A) 19/128 (B) 23/128 (C) 1/4 (D) 35/128 (E) 1/2

Solution: The number of coins on both heads could be 0, 1, 2, or 3. The probability that we obtain 0 heads with both coins is

$$\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^7.$$

The probability that we obtain exactly 1 head with both coins is

$$3\left(\frac{1}{2}\right)^3 \cdot 4\left(\frac{1}{2}\right)^4 = 12\left(\frac{1}{2}\right)^7.$$

The probability that we obtain exactly 2 heads with both coins is

$$3\left(\frac{1}{2}\right)^3 \cdot 6\left(\frac{1}{2}\right)^4 = 18\left(\frac{1}{2}\right)^7.$$

The probability that we obtain exactly 3 head with both coins is

$$\left(\frac{1}{2}\right)^3 \cdot 4\left(\frac{1}{2}\right)^4 = 4\left(\frac{1}{2}\right)^7.$$

Thus, the desired probability is

$$(1 + 12 + 18 + 4)\left(\frac{1}{2}\right)^7 = \frac{35}{128}.$$

Example 10.6 The Monty Hall Problem. Consider the following game. Three boxes are marked A, B, and C, and one of them contains \$1,000,000. You choose box C. I'm going to help you out now. I tell you that box A does not contain the money. Now, you decide whether you want to change and choose box B, or keep box C.

Solution: At first glance, you might say that the money is equally likely to be in box B or C, so changing doesn't help at all. However, consider the probability of winning this game. If you never change, the only way that you win is if you choose the right box first, a 1/3 chance. If you change instead, you will always win if you pick a wrong box first, because after I expose one wrong box, the other unchosen box is a winner. Since you have a 2/3 chance of picking the wrong box initially, you have a 2/3 chance of winning if you accept the offer to switch boxes.

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11 Statistics

Most of the AMC statistics problems involve the concepts of mean (average), median, and mode.

Definition 11.1 The **(arithmetic) mean** of a collection of n numbers $a_1,\ a_2, \dots, a_n$ is

$$\text{mean} = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Definition 11.2 Consider the collection of numbers $\{a_1,\ a_2, \dots, a_n\}$, with $a_1 \leq a_2 \leq \cdots \leq a_n$. The **median** of the collection is the middle term of the collection when n is odd and is the average of the two middle terms when n is even.

Definition 11.3 The **mode** of a collection of n numbers $a_1,\ a_2, \dots, a_n$ is the term(s) that occur most frequently.

Definition 11.4 The **range** of a collection of n integers $a_1,\ a_2, \dots, a_n$ is the difference between the largest and smallest integers.

Example 11.1 (1991 AHSME #16) One hundred students at Century High School participated in the AMC 12 last year, and their mean score was 100. The number of non-seniors taking the AMC 12 was 50% more than the number of seniors, and the mean score of the seniors was 50% higher than that of non-seniors. What was the mean score of the seniors?

(A) 100 (B) 112.5 (C) 120 (D) 125 (E) 150

Solution: Let S denote the number of seniors that participated in the contest. Then the number of non-seniors that participated in the contest is $1.5S$, and

$$S + 1.5S = 100, \text{ so } S = 40.$$

Thus there are 40 seniors and 60 non-seniors that participated in the contest. Let M be the mean of the seniors. Then the mean of the non-seniors is $\frac{2}{3}M$. The sum of the seniors' scores is $40M$ and the sum of the non-seniors' scores is $60 \cdot \frac{2}{3}M = 40M$. Thus, we obtain

$$100 = \frac{40M + 40M}{100},$$

so

$$M = 125.$$

Example 11.2 (2004 AMC 12A #10) The sum of 49 consecutive integers is 7^5. What is their median?

(A) 7 (B) 7^2 (C) 7^3 (D) 7^4 (E) 7^5

Solution. Since the integers are consecutive, the mean and the median are the same with value

$$\text{median} = \text{mean} = \frac{7^5}{49} = \frac{7^5}{7^2} = 7^3.$$

Example 11.3 (2000 AMC 12 #9) Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order onto a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walter entered?

(A) 71 (B) 76 (C) 80 (D) 82 (E) 91

Solution: Since the average of the first two scores entered is an integer, the first two scores entered must both be even or both be odd. If they were the two odd scores, then their sum would be $71 + 91 = 162$. Since 162 is divisible by 3, the third score added would also have to be divisible by 3 so that the average of the first three scores is an integer. However, none of the remaining scores (76, 80, and 82) is divisible by 3, so the first two scores that she entered could not have been the odd scores.

Thus, the first two scores entered must have both been even. Since

$$76 + 80 = 156 \text{ and } 80 + 82 = 162$$

are both divisible by 3, but none of 71, 91, and 82 is divisible by 3, the first pair chosen could not have been 76 and 80 or 80 and 82. Thus, the first two scores entered must have been 76 and 82. Since $76 + 82 = 158$ has a remainder of 2 when divided by 3, the third score entered must have a remainder of 1 when divided by 3. Of those remaining, only 91 has this property. So the first three scores entered must have been 76, 82, and 91. Since the sum of 76, 82, and 91 is odd, the fourth score entered must also be odd. Thus, the fourth score entered was 71 and the last score entered was 80.

12 Sequences and Series

Definition 12.1 A **sequence** of numbers is a function that assigns to each positive integer a distinct number. The number assigned by the sequence to the integer n is commonly denoted using a subscript, such as a_n. These numbers are called the **terms** of the sequence. You should think of a sequence as a *list* of numbers.

Example 12.1 Consider the sequence of numbers

$$1,\ 4,\ 7, \ldots.$$

We have $a_1 = 1$, $a_2 = 4$, $a_3 = 7$, and in general (assuming that the pattern continues), $a_n = 3n - 2$.

There are two special types of sequences that arise frequently on the AMC competitions.

Definition 12.2 An **arithmetic sequence** (also called an *arithmetic progression*) $\{a_n\}$ is defined by

$$a_n = a + (n-1)d$$

for some constants a and $d \neq 0$. The number a is the first term of the sequence and the number d is the common difference between terms.

Example 12.2 The sequence

$$1,\ 4,\ 7, \ldots$$

that we considered previously is an arithmetic sequence with first term $a = 1$ and common difference $d = 3$.

Theorem 12.1 If $\{a_n\}$ is an arithmetic sequence with first term a and common difference d, then

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{n}{2}(a_1 + a_n).$$

Definition 12.3 A **geometric sequence** $\{a_n\}$ is defined by

$$a_n = a \cdot r^{n-1}$$

where a is a constant and r is a constant not equal to 0 or 1. The number a is the first term of the sequence and the number r is the common ratio between terms.

Example 12.3 The sequence

$$2,\ 6,\ 18, \ldots$$

is a geometric sequence with first term $a = 2$ and common ratio $r = 3$.

Theorem 12.2 If $\{a_n\}$ is a geometric sequence with first term a and common ratio r, then

$$a_1 + a_2 + \cdots + a_n = a\frac{1-r^n}{1-r}.$$

Proof. Let S_n denote the sum of the first n terms. Then:

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rS_n &= ar + ar^2 + ar^3 + \cdots + ar^n \\ S_n - rS_n &= a - ar^n. \end{aligned}$$

Definition 12.4 A **recursively-defined sequence** is one in which the n-th term a_n is defined in terms of the previous terms $a_1, a_2, \ldots, a_{n-1}$.

Example 12.4 The *Fibonacci sequence* is an example of a recursively-defined sequence:

$$f_n = f_{n-1} + f_{n-2}, \quad f_1 = 0, \quad f_2 = 1.$$

The first few terms of the Fibonacci sequence are 0,1,1,2,3,5,8,13,....

Example 12.5 (2002 AMC 12B #13) The sum of 18 consecutive positive integers is a perfect square. What is the smallest possible value of this sum?

(A) 169 (B) 225 (C) 289 (D) 361 (E) 441

Solution: Let a denote the first term of the sequence. The sequence is an arithmetic sequence with $d = 1$, and the sum is

$$\frac{18}{2}a + (a+17) = 9(2a+17).$$

For the sum to be a perfect square, the term $2a + 17$ must be a perfect square. This first occurs when $a = 4$, which gives the sum 225.

Example 12.6 (1981 AHSME #14) In a geometric sequence of real numbers, the sum of the first two terms is 7 and the sum of the first six terms is 91. What is the sum of the first four terms?

(A) 28 (B) 32 (C) 35 (D) 49 (E) 84

Solution: Let a denote the first term of the sequence and let r denote the common ratio. We have

$$7 = a + ar \text{ and } 91 = 7(1 + r^2 + r^4).$$

Thus $r^2 = 3$ and the sum of the first four terms is 28.

Example 12.7 (2004 AMC 10A #24) Let $a_1, a_2, \ldots$ be a sequence with the following properties:

- $a_1 = 1$, and
- $a_{2n} = n \cdot a_n$ for any positive integer n.

What is the value of $a_{2^{100}}$?

(A) 1 (B) 2^{99} (C) 2^{100} (D) 2^{4950} (E) 2^{9999}

Solution: Computing the first few terms, we obtain:

$$\begin{aligned}
a_2 &= a_{2\cdot 1} = 1 \cdot a_1 = 1 = 2^0 \\
a_{2^2} &= a_4 = a_{2\cdot 2} = 2 \cdot a_2 = 2 = 2^1 \\
a_{2^3} &= a_8 = a_{2\cdot 4} = 4 \cdot a_4 = 8 = 2^{1+2} \\
a_{2^4} &= a_{16} = a_{2\cdot 8} = 8 \cdot a_8 = 64 = 2^{1+2+3}
\end{aligned}$$

In general, we observe that

$$a_{2^n} = 2^{1+2+\cdots+(n-1)} = 2^{(n-1)n/2}.$$

Thus, $a_{2^{100}} = 2^{4950}$.

13 Prime Factorization

Definition 13.1 A natural number greater than 1 is said to be **prime** if its only natural number divisors are 1 and itself. Natural numbers greater than 1 that are not prime are **composite**.

Theorem 13.1 The Fundamental Theorem of Arithmetic. Every natural number, other than 1, can be factored into a product of primes in only one way, apart from the order of the factors.

Example 13.1 Find positive integers x and y that satisfy both

$$xy = 40 \text{ and } 31 = 2x + 3y.$$

Solution: Since 40 has the unique factorization $40 = 2^3 \cdot 5$, there are only 8 possibilities for the pair (x, y). These are $(1, 40), (2, 20), (4, 10), (8, 5), (5, 8), (10, 4), (20, 2), (40, 1)$. Only $(x, y) = (8, 5)$ additionally satisfies $31 = 2x + 3y$.

Example 13.2 (1998 AHSME #6) Suppose that 1998 is written as a product of two positive integers whose difference is as small as possible. What is this difference?

(A) 8 (B) 15 (C) 17 (D) 47 (E) 93

Solution: The prime decomposition of 1998 is $1998 = 2 \cdot 3^3 \cdot 37$. The factorization of 1998 into two positive integers whose difference is as small as possible is $1998 = 37 \cdot 54$ and the difference is $54 - 37 = 17$.

Example 13.3 (2005 AMC 10A #15) How many positive cubes divide $3! \cdot 5! \cdot 7!$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution: We have $3! \cdot 5! \cdot 7! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. There are 6 distinct cubes that divide $3! \cdot 5! \cdot 7!$.

Example 13.4 (2001 AMC 12 #21) The product of four positive integers a, b, c, d is 8! and they satisfy the equations

$$\begin{aligned} ab + a + b &= 524 \\ bc + b + c &= 146 \\ cd + c + d &= 104. \end{aligned}$$

What is $a - d$?

(A) 4 (B) 6 (C) 8 (D) 10 (E) 12

Solution: We can rewrite the given equations as

$$\begin{aligned} ab+a+b+1 &= 525=(a+1)(b+1)=3\cdot 5^2\cdot 7 \\ bc+b+c+1 &= 147=(b+1)(c+1)=3\cdot 7^2 \\ cd+c+d+1 &= 105=(c+1)(d+1)=3\cdot 5\cdot 7. \end{aligned}$$

Since $(a+1)(b+1)$ has a factor of $5^2=25$, but $(b+1)(c+1)$ has no factor of 5, $(a+1)$ must be divisible by 25. Similarly, $(d+1)$ must be divisible by 5. Since $(b+1)(c+1)=3\cdot 7^2$, either $(b+1)=7$ and $(c+1)=3\cdot 7$ or $(b+1)=3\cdot 7$ and $(c+1)=7$. If $(b+1)=7$, then $(a+1)=3\cdot 25=75$, but 8! is not divisible by 75. Thus, $(b+1)=3\cdot 7$ and $(c+1)=7$. Thus $a=24$, $b=20$, $c=6$, and $d=14$, so $a-d=10$.

Example 13.5 Find the smallest positive integer n such that $n/2$ is a perfect square, $n/3$ is a perfect cube, and $n/5$ is a perfect fifth power.

Solution: Since n is divisible by 2, 3, and 5, we may assume that it has the form $n=2^a3^b5^c$. Then

$$\begin{aligned} n/2 &= 2^{a-1}3^b5^c \\ n/3 &= 2^a3^{b-1}5^c \\ n/5 &= 2^a3^b5^{c-1}. \end{aligned}$$

Since $n/2$ must be a perfect square, $a-1$, b, and c must all be even. Since $n/3$ is a perfect cube, a, $b-1$, and c must all be multiples of 3. Since $n/5$ is a perfect fifth power, a, b, and $c-1$ must all be multiples of 5. The smallest values that satisfy these conditions are $a=15$, $b=10$, and $c=6$. Thus, $n=2^{15}3^{10}5^6$ is the smallest such positive integer.

Example 13.6 Show that $\log_{10} 2$ is irrational.

Solution: If $\log_{10} 2$ is rational, then there exist integers r, s such that

$$\log_{10} 2=\frac{r}{s}.$$

Then

$$10^{r/s}=2,$$

so

$$10^r=2^s,$$

or

$$5^r 2^r = 2^s,$$

which contradicts the FTA. Thus, $\log_{10} 2$ is irrational.

Example 13.7 Find all positive integers n such that $2^8+2^{11}+2^n$ is a perfect square.

Solution: Suppose that k is an integer such that

$$k^2 = 2^8 + 2^{11} + 2^n = 2304 + 2^n = 48^2 + 2^n.$$

Then

$$k^2 - 48^2 = (k-48)(k+48) = 2^n.$$

By the FTA,

$$k - 48 = 2^s \text{ and } k + 48 = 2^t,$$

where $s + t = n$. But then

$$2^t - 2^s = 48 - (-48) = 96 = 3 \cdot 2^5,$$

so

$$2^s(2^{t-s} - 1) = 3 \cdot 2^5.$$

By the FTA, $s = 5$ and $t - s = 2$, so $s + t = n = 12$. Thus, the only natural number n such that $2^8 + 2^{11} + 2^n$ is a perfect square is $n = 12$.

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14 Number Bases

14.1 The Decimal (Base-10) Scale.

Any natural number n can be written in the form

$$n = a_0 10^k + a_1 10^{k-1} + a_2 10^{k-2} + \cdots + a_{k-1} 10 + a_k$$

where $1 \le a_0 \le 9, 0 \le a_j \le 9, j \ge 1$. This is the *decimal*, or *base-10* representation of n. For example

$$65789 = 6 \cdot 10^4 + 5 \cdot 10^3 + 7 \cdot 10^2 + 8 \cdot 10 + 9.$$

Example 14.1 Find a reduced fraction equivalent to the repeating decimal $0.\overline{123} = 0.123123123\ldots$.

Solution: Let $N = 0.123123123\ldots$. Then $1000N = 123.123123123\ldots$. Hence $1000N - N = 123$, so

$$N = \frac{123}{999} = \frac{41}{333}.$$

Example 14.2 What are all the two-digit positive integers in which the difference between the integer and the product of its two digits is 12?

Solution: Let such an integer be $10a+b$, where a, b are digits. Solving $10a+b-ab = 12$ for a, we obtain

$$a = \frac{12-b}{10-b} = 1 + \frac{2}{10-b}.$$

Since a is an integer, $10 - b$ must be a positive integer that divides 2. This gives $b = 8, a = 2$ or $b = 9, a = 3$. Thus, 28 and 39 are the only such integers.

Example 14.3 Find all the integers with initial digit 6 such that if this initial integer is suppressed, the resulting number is 1/25 of the original number.

Solution: Let x be the integer sought. Then $x = 6 \cdot 10^n + y$ where y is a positive integer. The given condition stipulates that

$$y = \frac{1}{25}\left(6 \cdot 10^n + y\right),$$

that is,

$$y = \frac{10^n}{4} = 25 \cdot 10^{n-2}.$$

This requires $n \ge 2$, whence $y = 25, 250, 2500, 25000$, etc.. Therefore $x = 625, 6250, 62500, 625000$, etc..

Example 14.4 (1986 IMO) Find all natural numbers x such that the product of their digits (in decimal notation) equals $x^2 - 10x - 22$.

Solution: Let x have the form

$$x = a_0 + a_1 10 + a_2 10^2 + \cdots + a_n 10^n,\ a_k \leq 9,\ a_n \neq 0.$$

Let $P(x)$ be the product of the digits of x,

$$P(x) = x^2 - 10x - 22.$$

Now

$$P(x) = a_0 a_1 \cdots a_n \leq 9^n a_n < 10^n a_n \leq x,$$

where strict inequality occurs when x has more than one digit. This means that $x^2 - 10x - 22 \leq x$ which implies that $x < 13$, so x has one digit or $x = 10, 11$ or 12. Since $x^2 - 10x - 22 = x$ has no integral solutions, x cannot have one digit. If $x = 10, P(x) = 0$, but $x^2 - 10x - 22 \neq 0$. If $x = 11, P(x) = 1$, but $x^2 - 10x - 22 \neq 1$. The only solution is thus $x = 12$.

Example 14.5 (1987 AIME) An ordered pair (m, n) of non-negative integers is called *simple* if the addition $m + n$ requires no carrying. Find the number of simple ordered pairs of non-negative integers that add to 1492.

Solution: Observe that there are $d+1$ solutions to $x+y = d$, where x, y are positive integers and d is a single-digit integer. These are

$$(0 + d),\ (1 + d - 1),\ (2 + d - 2), \ldots,\ (d + 0)$$

Since there is no carrying, we search for the numbers of solutions of this form to $x + y = 1$, $u + v = 4$, $s + t = 9$, and $a + b = 2$. Since each separate solution may combine with any other, the total number of simple pairs is

$$(1 + 1)(4 + 1)(9 + 1)(2 + 1) = 300.$$

Example 14.6 (1992 AIME) For how many pairs of consecutive integers in

$$\{1000, 1001, \ldots, 2000\}$$

is no carrying required when the two integers are added?

Solution: Other than 2000, a number on this list has the form $n = 1000 + 100a + 10b + c$, where a, b, c are digits. If there is no carrying in adding n and $n + 1$ then n has the form

$$1999,\ 1000 + 100a + 10b + 9,\ 1000 + 100a + 99, 1000 + 100a + 10b + c$$

with $0 \leq a, b, c \leq 4$, i.e., five possible digits. There are $5^3 = 125$ integers of the form $1000 + 100a + 10b + c, 0 \leq a, b, c \leq 4$, $5^2 = 25$ integers of the form $1000 + 100a + 10b + 9, 0 \leq a, b \leq 4$, and 5 integers of the form $1000 + 100a + 99, 0 \leq a \leq 4$. The total of integers sought is thus $125 + 25 + 5 + 1 = 156$.

Example 14.7 (1992 AIME) Let S be the set of all rational numbers r, $0 < r < 1$, that have a repeating decimal expansion of the form

$$0.abcabcabc\ldots = 0.\overline{abc},$$

where the digits a, b, c are not necessarily distinct. To write the elements of S as fractions in lowest terms, how many different numerators are required?

Solution: Observe that $0.abcabcabc\ldots = \frac{abc}{999}$, and that $999 = 3^3 \cdot 37$. If abc is neither divisible by 3 nor by 37, the fraction is already in lowest terms. By Inclusion-Exclusion there are

$$999 - \left(\frac{999}{3} + \frac{999}{37}\right) + \frac{999}{3 \cdot 37} = 648$$

such fractions. Also, fractions of the form $\frac{s}{37}$ where s is divisible by 3 but not by 37 are in S. There are 12 fractions of this kind (with s = 3, 6, 9, 12, ..., 36). We do not consider fractions of the form $\frac{l}{3^t}, t \leq 3$ with l divisible by 37 but not by 3, because these fractions are > 1 and hence not in S. The total number of distinct numerators in the set of reduced fractions is thus $640 + 12 = 660$.

14.2 Non-decimal Scales

Given any positive integer $r > 1$, we can express any number x in base r. If n is a positive integer, and $r > 1$ is an integer, then n has the base-r representation

$$n = a_0 + a_1 r + a_2 r^2 + \cdots + a_k r^k,\ 0 \leq a_t \leq r - 1,\ a_k \neq 0,\ r^k \leq n < r^{k+1}.$$

We use the convention that we shall refer to a decimal number (i.e. a base-10 number) without referring to its base, and to a base-r number by using the subindex $_r$.

Definition 14.1 We will refer to the base-2 representation of a number as the **binary** form of the number. We will refer to the base-3 representation of a number as the **ternary** form of the number.

Example 14.8 Express the decimal number 5213 in base-7.

Solution: Observe that $5213 < 7^5$. We thus want to find $0 \leq a_0, \ldots, a_4 \leq 6, a_4 \neq 0$ such that

$$5213 = a_4 7^4 + a_3 7^3 + a_2 7^2 + a_1 7 + a_0.$$

Dividing by 7^4, we obtain $2+$ proper fraction $= a_4+$ proper fraction. This means that $a_4 = 2$. Thus

$$5213 = 2 \cdot 7^4 + a_3 7^3 + a_2 7^2 + a_1 7 + a_0 = 4802 + a_3 7^3 + a_2 7^2 + a_1 7 + a_0.$$

Thus,

$$411 = a_3 7^3 + a_2 7^2 + a_1 7 + a_0.$$

Dividing by 7^3, we obtain 1+ proper fraction $= a_3+$ proper fraction, and so $a_3 = 1$. Continuing in this way we find that

$$5213 = 2 \cdot 7^4 + 1 \cdot 7^3 + 1 \cdot 7^2 + 2 \cdot 7 + 5 = 21125_7.$$

Example 14.9 Rewrite the base-6 number 3425_6 in base 8.

Solution: First, we rewrite the base-6 number 3425_6 in base-10 representation:

$$\begin{aligned} 3425_6 &= 3 \cdot 6^3 + 4 \cdot 6^2 + 2 \cdot 6 + 5 \\ &= 809. \end{aligned}$$

Since $8^2 = 64$ and $8^3 = 512$, we obtain:

$$\begin{aligned} 3425_6 &= 809 \\ &= 1 \cdot 8^3 + +4 \cdot 8^2 + 5 \cdot 8 + 3 \\ &= 1453_8. \end{aligned}$$

Example 14.10 Write 562_7 in base-5.

Solution: $562_7 = 5 \cdot 7^2 + 6 \cdot 7 + 2 =$ in decimal scale, so the problem reduces to convert 289 to base-five:

$$289 = 2 \cdot 5^3 + 1 \cdot 5^2 + 2 \cdot 5 + 4,$$

so

$$562_7 = 289 = 2124_5.$$

Example 14.11 (1981 AHSME #16) The base-3 representation of x is

$$12112211122211112222_3.$$

What is the first digit (on the left) of the base-9 representation of x?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Since $9 = 3^2$, we will group the base-3 digits in pairs, starting from the right. Then

$$\begin{aligned} x &= (12)(11)(22)(11)(12)(22)(11)(11)(22)(22)_3 \\ &= (1\cdot 3+2)\cdot 3^{18} + (1\cdot 3+1)\cdot 3^{16} + (2\cdot 3+2)\cdot 3^{14} + (1\cdot 3+1)\cdot 3^{12} \\ &\quad +(1\cdot 3+2)\cdot 3^{10} + (2\cdot 3+2)\cdot 3^{8} + (1\cdot 3+1)\cdot 3^{6} + (1\cdot 3+1)\cdot 3^{4} \\ &\quad +(2\cdot 3+2)\cdot 3^{2} + (2\cdot 3+2) \\ &= 5\cdot 9^8 + 4\cdot 9^7 + 8\cdot 9^6 + 4\cdot 9^5 + 5\cdot 9^4 + 4\cdot 9^3 + 4\cdot 9^2 + 8\cdot 9^1 + 8 \\ &= 5484584488_9. \end{aligned}$$

Example 14.12 (1986 AIME) The increasing sequence

$$1, 3, 4, 9, 10, 12, 13, \ldots$$

consists of all those positive integers which are powers of 3 or sums of distinct powers or 3. Find the hundredth term of the sequence.

Solution: If the terms of the sequence are written in base-three, they comprise the positive integers which do not contain the digit 2. Thus the terms of the sequence in ascending order are

$$1_3, 10_3, 11_3, 100_3, 101_3, 110_3, 111_3, \ldots$$

In the *binary* scale these numbers are, of course, the ascending natural numbers $1, 2, 3, 4, \ldots$. Therefore to obtain the 100th term of the sequence we write 100 in binary and then translate this into ternary: $100 = 1100100_2$ and $1100100_3 = 3^6 + 3^5 + 3^2 = 981$.

15 Modular Arithmetic

Definition 15.1 Given a positive integer n, we say that the integer a is equal to the integer b **modulo** n, written

$$a \equiv b \mod n$$

if n divides $a - b$. Equivalently, $a \equiv b \mod n$ if b is the remainder that results when a is divided by n.

For example,

$$37 \equiv 2 \mod 5 \text{ and } 9 \equiv 1 \mod 4.$$

Note also that

$$-11 \equiv 4 \mod 3$$

since $-11 - 4 = -15$ is divisible by 3.

Since $n|(a-b)$ implies that there is an integer k such that $nk = a - b$, we deduce that $a \equiv b \mod n$ if and only if there is an integer k such that $a = b + nk$.

Theorem 15.1 Let $n \geq 2$ be an integer. If $x \equiv y \mod n$ and $u \equiv v \mod n$ then

$$ax + bu \equiv ay + bv \mod n.$$

Proof. Since $n|(x-y)$ and $n|(u-v)$, there are integers s and t such that $ns = x - y$ and $nt = u - v$. This implies that

$$a(x-y) + b(u-v) = n(as + bt),$$

which in turn implies that

$$n|(ax + bu - ay - bv).$$

Thus

$$ax + bu \equiv ay + bv \mod n.$$

Corollary 15.2 Let $n \geq 2$ be an integer. If $x \equiv y \mod n$ and $u \equiv v \mod n$ then

$$xu \equiv yv \mod n.$$

Proof. Let $a = u, b = y$ in Theorem 15.1.

Corollary 15.3 Let $n > 1$ be an integer, $x \equiv y \mod n$ and j a positive integer. Then $x^j \equiv y^j \mod n$.

Proof. Use Corollary 15.2 repeatedly with $u = x, v = y$.

Corollary 15.4 Let $n > 1$ be an integer, $x \equiv y \mod n$. If f is a polynomial with integral coefficients then $f(x) \equiv f(y) \mod n$.

Example 15.1 Find the units digit of 7^{100}.

Solution: To find the units digit of 7^{100}, we must find 7^{100} modulo 10.

$$\begin{aligned} 7^2 &\equiv -1 \mod 10 \\ 7^3 &\equiv 7 \cdot 7^2 \mod 10 \\ &\equiv -7 \mod 10 \\ 7^4 &\equiv (7^2)^2 \mod 10 \\ &\equiv (-1)^2 \mod 10 \\ &\equiv 1 \mod 10 \\ 7^{100} &\equiv (7^4)^{25} \mod 10 \\ &\equiv 1^{25} \mod 10 \\ &\equiv 1 \mod 10. \end{aligned}$$

Thus, the units digit of 7^{100} is 1.

Example 15.2 Find the remainder when 6^{1987} is divided by 37.

Solution: $6^2 \equiv -1 \mod 37$. Thus

$$6^{1987} \equiv 6 \cdot 6^{1986} \equiv 6(6^2)^{993} \equiv 6(-1)^{993} \equiv -6 \equiv 31 \mod 37$$

and the remainder sought is 31.

Example 15.3 Find the remainder when

$$12233 \cdot 455679 + 87653^3$$

is divided by 4.

Solution: $12233 = 12200 + 32 + 1 \equiv 1 \mod 4$. Similarly, $455679 = 455600 + 76 + 3 \equiv 3$, $87653 = 87600 + 52 + 1 \equiv 1 \mod 4$. Thus

$$12233 \cdot 455679 + 87653^3 \equiv 1 \cdot 3 + 1^3 \equiv 4 \equiv 0 \mod 4.$$

This means that $12233 \cdot 455679 + 87653^3$ is divisible by 4.

Example 15.4 Prove that 7 divides $3^{2n+1} + 2^{n+2}$ for all natural numbers n.

Solution: Observe that

$$3^{2n+1} \equiv 3 \cdot 9^n \equiv 3 \cdot 2^n \mod 7$$

and

$$2^{n+2} \equiv 4 \cdot 2^n \mod 7$$

. Hence

$$3^{2n+1} + 2^{n+2} \equiv 7 \cdot 2^n \equiv 0 \mod 7,$$

for all natural numbers n.

Example 15.5 Prove the following result of Euler: $641|(2^{32} + 1)$.

Solution: Observe that $641 = 2^7 \cdot 5 + 1 = 2^4 + 5^4$. Hence $2^7 \cdot 5 \equiv -1 \mod 641$ and $5^4 \equiv -2^4 \mod 641$. Now, $2^7 \cdot 5 \equiv -1 \mod 641$ yields

$$5^4 \cdot 2^{28} = (5 \cdot 2^7)^4 \equiv (-1)^4 \equiv 1 \mod 641.$$

This last congruence and

$$5^4 \equiv -2^4 \mod 641$$

yield

$$-2^4 \cdot 2^{28} \equiv 1 \mod 641,$$

which means that $641|(2^{32} + 1)$.

Example 15.6 Prove that $7|(2222^{5555} + 5555^{2222})$.

Solution: $2222 \equiv 3 \mod 7$, $5555 \equiv 4 \mod 7$ and $3^5 \equiv 5 \mod 7$. Now

$$2222^{5555} + 5555^{2222} \equiv 3^{5555} + 4^{2222} \equiv (3^5)^{1111} + (4^2)^{1111} \equiv 5^{1111} - 5^{1111} \equiv 0 \mod 7.$$

Example 15.7 Find the units digit of 7^{7^7}.

Solution: We must find $7^{7^7} \mod 10$. Now, $7^2 \equiv -1 \mod 10$, and so $7^3 \equiv 7^2 \cdot 7 \equiv -7 \equiv 3 \mod 10$ and $7^4 \equiv (7^2)^2 \equiv 1 \mod 10$. Also, $7^2 \equiv 1 \mod 4$ and so $7^7 \equiv (7^2)^3 \cdot 7 \equiv 3 \mod 4$, which means that there is an integer t such that $7^7 = 3 + 4t$. Upon assembling all this,

$$7^{7^7} \equiv 7^{4t+3} \equiv (7^4)^t \cdot 7^3 \equiv 1^t \cdot 3 \equiv 3 \mod 10.$$

Thus the last digit is 3.

Example 15.8 Find infinitely many integers n such that $2^n + 27$ is divisible by 7.

Solution: Observe that $2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 1, 2^4 \equiv 2, 2^5 \equiv 4, 2^6 \equiv 1 \mod 7$ and so $2^{3k} \equiv 1 \mod 3$ for all positive integers k. Hence $2^{3k} + 27 \equiv 1 + 27 \equiv 0 \mod 7$ for all positive integers k. This produces the infinitely many values sought.

Example 15.9 Prove that $2^k - 5, k = 0, 1, 2, \ldots$ never leaves remainder 1 when divided by 7.

Solution: $2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 1 \mod 7$, and this cycle of three repeats. Thus $2^k - 5$ can leave only remainders 3, 4, or 6 upon division by 7.

Example 15.10 (1994 AIME) The increasing sequence

$$3, 15, 24, 48, \ldots,$$

consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 1994-th term of the sequence is divided by 1000?

Solution: The sequence consists of numbers $n^2 - 1$ such that

$$3|(n^2 - 1) = (n - 1)(n + 1).$$

Since 3 is prime, this implies that

$$3 \mid (n - 1) \text{ or } 3 \mid (n + 1).$$

Thus,

$$n = 3k + 1 \text{ or } n = 3k - 1,$$

where $k = 1, 2, 3, \ldots$. The sequence

$$3k + 1, \quad k = 1, 2, \ldots$$

produces the terms

$$n^2 - 1 = (3k + 1)^2 - 1 = 15, 48, \ldots,$$

which are the terms at even places of the sequence of $3, 15, 24, 48, \ldots$. The sequence

$$3k - 1, k = 1, 2, \ldots$$

produces the terms

$$n^2 - 1 = (3k - 1)^2 - 1 = 3, 24, \ldots,$$

which are the terms at odd places of the sequence $3, 15, 24, 48, \ldots$. We must find the 997th term of the sequence $3k + 1, k = 1, 2, \ldots$. Thus, the term sought is $(3(997) + 1)^2 - 1 \equiv (3(-3) + 1)^2 - 1 \equiv 8^2 - 1 \equiv 63 \mod 1000$. The remainder sought is thus 63.

Example 15.11 Find the remainder when 3^{2006} is divided by 8.

Solution: Since $3^2 = 9$, we have $3^2 \equiv 1 \mod 8$. This implies that

$$\begin{aligned} 3^{2006} &\equiv \left(3^2\right)^{1003} \mod 8 \\ &\equiv 1^{1003} \mod 8 \\ &\equiv 1 \mod 8. \end{aligned}$$

Thus, the remainder is 1.

Example 15.12 Find the remainder when 3^{2006} is divided by 11.

Solution: First, note that $3^5 = 243 \equiv 1 \mod 11$. Thus

$$\begin{aligned} 3^{2006} &\equiv 3^{2005} \cdot 3 \mod 11 \\ &\equiv \left(3^5\right)^{401} \cdot 3 \mod 11 \\ &\equiv 1^{401} \cdot 3 \mod 11 \\ &\equiv 3 \mod 11. \end{aligned}$$

Thus, the remainder is 3.

Example 15.13 (1999 AHSME #25) There are unique integers $a_2, a_3, a_4, a_5, a_6, a_7$ such that

$$\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!},$$

where $0 \leq a_i \leq i$ for $i = 2, 3, 4, 5, 6, 7$. What is $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Solution: First, multiply both sides of the equation by 7! to clear the denominators:

$$5 \cdot 6! = a_2(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) + a_3(7 \cdot 6 \cdot 5 \cdot 4) + a_4(7 \cdot 6 \cdot 5) + a_5(7 \cdot 6) + a_6 \cdot + a_7.$$

Consider this equation modulo 7. We obtain

$$a_7 \equiv 5 \cdot 6! \mod 7 \equiv 2 \mod 7.$$

Since $0 \leq a_7 > 7$, we conclude that $a_7 = 2$. Next, consider the equation modulo 6. We obtain $a_6 = 4$. Continuing, we obtain $a_5 = 0$ and $a_4 = a_3 = a_2 = 1$. Thus, $a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 9$.

Example 15.14 Prove that a sum of two squares of integers leaves remainder 0, 1 or 2 when divided by 4.

Solution: An integer is either even (of the form $2k$) or odd (of the form $2k+1$). We have

$$\begin{aligned}(2k)^2 &= 4(k^2),\\ (2k+1)^2 &= 4(k^2+k)+1.\end{aligned}$$

Thus squares leave remainder 0 or 1 when divided by 4 and hence their sum leave remainder 0, 1, or 2.

Example 15.15 Prove that 2003 is not the sum of two squares by proving that the sum of any two squares cannot leave remainder 3 upon division by 4.

Solution: 2003 leaves remainder 3 upon division by 4. But we know from example 15.14 that sums of squares do not leave remainder 3 upon division by 4, so it is impossible to write 2003 as the sum of squares.

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16 Integer Division Results

Using modular arithmetic, we obtain the following techniques for determining certain integer factors of numbers.

Theorem 16.1 Theorem: Integer Division Results. Let n be a positive integer. Then:

- n is divisible by 2 if and only if the units digit of n is even.
- n is divisible by 3 if and only if the sum of its digits is divisible by 3.
- n is divisible by 5 if and only if the units digit of n is a 0 or a 5.
- n is divisible by 7 if and only if 7 divides the integer that results from first truncating n by removing its units digit, and then subtracting twice the value of this digit from the truncated integer.
- n is divisible by 9 if and only if the sum of its digits is divisible by 9.
- n is divisible by 11 if and only if the alternating (by positive and negative signs) sum of its digits is divisible by 11.

Proof.

- The division results by 2 and 5 should be clear.
- Let's consider the situation of division by 3. Suppose that the positive integer n is written in its decimal expansion as

  $$n = a_k \cdot 10^k + a_{k-1}10^{k-1} + \cdot + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0.$$

  Since $10 \equiv 1 \mod 3$, for any positive integer j we have $10^j \equiv 1^j \equiv 1 \mod 3$. Thus,

  $$n \equiv (a_k + a_{k-1} + \cdots + a_2 + a_1 + a_0) \mod 3.$$

  So the remainder when n is divisible by 3 is the same as the remainder when the sum of the digits is divisible by 3. This remainder is 0 if and only if 3 divides n.
- Since $10 \equiv 1 \mod 9$, and $10^j \equiv 1^j \equiv 1 \mod 9$, the proof for divisibility by 9 is analogous to the proof for divisibility by 3.
- The divisibility by 11 result follows from the observation that

  $$10 \equiv -1 \mod 11,$$

  so for any positive integer j, we have

  $$10^j \equiv (-1)^j \mod 11.$$

- Finally, we'll consider the divisibility by 7 results. First, we can show that the result is always true provided that it is true for all integers with at most 6 digits, i.e. when

$$n = a_5 \cdot 10^5 + a_4 \cdot 10^4 + a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0.$$

Then we note that since $10 \equiv 3 \mod 7$, we have $10^2 \equiv 2 \mod 7$, $10^3 \equiv 6 \mod 7$, $10^4 \equiv 4 \mod 7$, and $10^5 \equiv 5 \mod 7$. Now suppose that $n \equiv b \mod 7$, where $0 \leq b \leq 6$. Then

$$n \equiv b \equiv (5a_5 + 4a_4 + 3a_3 + 2a_2 + 3a_1 + a_0) \mod 7,$$

so that

$$a_0 \equiv (b - (5a_5 + 4a_4 + 3a_3 + 2a_2 + 3a_1)) \mod 7.$$

Now consider the reduced number, which has the form

$$n_1 = a_5 \cdot 10^4 + a_4 \cdot 10^3 + a_3 \cdot 10^2 + a_2 \cdot 10 + a_1 - 2a_0.$$

To see that 7 also divides n_1 if and only if $b = 0$, note that

$$\begin{aligned} n_1 &\equiv (4a_5 + 6a_4 + 2a_3 + 3a_2 + a_1) \mod 7 \\ &\equiv (4a_5 + 6a_4 + 2a_3 + 3a_2 + a_1 - 2b + 2(5a_5 + 4a_4 + 6a_3 + 2a_2 + 3a_1) \mod 7 \\ &\equiv 14a_5 + 14a_4 + 14a_3 + 7a_2 + 7a_1 - 2b \mod 7 \\ &\equiv -2b \mod 7. \end{aligned}$$

Now, 7 divides n if and only if $b = 0$. But $b = 0$ if and only if $-2b = 0$, which implies that 7 divides n if and only if 7 divides n_1.

As an example of this result, consider the integer $n = 2233$. The reduced integer is

$$n_1 = 223 - 2 \cdot 3 = 217,$$

and the reduced integer for 217 is

$$n_2 = 21 - 2 \cdot 7 = 7.$$

Since 7 divides 7, 7 also divides 217, which implies that 7 also divides 2233.

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17 The Pigeonhole Principle

Theorem 17.1 The Pigeonhole Principle. If $n+1$ objects (e.g. pigeons) are to be placed in n boxes (e.g. pigeonholes), then there must be at least one box that contains more than one of the objects.

Note: This apparently trivial principle is very powerful. For example, it can be used to prove that in any group of 13 people, there are always two who have their birthday on the same month, and if the average human head has two million hairs, there are at least three people in NYC with the same number of hairs on their head. The Pigeonhole Principle is frequently useful for problems in which we need to determine a minimal number of objects to ensure that some integral number property is satisfied.

Theorem 17.2 The Extended Pigeonhole Principle. If more than mk objects are to be placed in k boxes, then at least one of the boxes must contain at least $m+1$ objects.

Example 17.1 Suppose that we draw cards consecutively and without replacement from a 52 card deck. How many do we need to draw to ensure that there will be a pair?

Solution: We can consider as boxes (the pigeonholes) the possible values of the cards (the pigeons). Then there are 13 boxes, 10 of which represent the cards numbered 1 (or Ace) through 10, and three additional boxes representing Jacks, Queens, and Kings. In order to guarantee that one box contains more than one card, we must draw $13+1=14$ total cards.

Example 17.2 Prove that among any set of 51 positive integers less than 100, there is a pair whose sum is 100.

Solution: Let $a_1, a_2, \ldots, a_{51}$ denote these 51 positive integers. Let $S_1 = \{1, 99\}$, $S_2 = \{2, 98\}$, $S_3 = \{3, 97\}, \ldots, S_k = \{k, 100-k\}, \ldots, S_{49} = \{49, 51\}, S_{50} = \{50\}$. Let $S_1, S_2, \ldots, S_{50}$ be the pigeonholes, and $a_1, a_2, \ldots, a_{51}$ the pigeons. Since there are 51 pigeons and 50 pigeonholes, there must be at least one pigeonhole with more than one pigeon. Thus, there must be at least one S_i with two numbers in it. Their sum is 100.

Example 17.3 Show that amongst any seven distinct positive integers not exceeding 126, one can find two of them, say a and b, which satisfy

$$b < a \leq 2b.$$

Solution: Split the numbers $\{1, 2, 3, \ldots, 126\}$ into the six sets

$$\{1, 2\}, \{3, 4, 5, 6\}, \{7, 8, \ldots, 13, 14\}, \{15, 16, \ldots, 29, 30\},$$

$$\{31, 32, \ldots, 61, 62\} \text{ and } \{63, 64, \ldots, 126\}.$$

By the Pigeonhole Principle, two of the seven numbers must lie in one of the six sets, and obviously, any such two will satisfy the stated inequality.

Example 17.4 Show that amongst any 55 integers from

$$\{1, 2, \ldots, 100\},$$

two must differ by 10.

Solution: First observe that if we choose $n + 1$ integers from any string of $2n$ consecutive integers, there will always be some two that differ by n. This is because we can pair the $2n$ consecutive integers

$$\{a + 1, a + 2, a + 3, \ldots, a + 2n\}$$

into the n pairs

$$\{a + 1, a + n + 1\}, \{a + 2, a + n + 2\}, \ldots, \{a + n, a + 2n\},$$

and if $n+1$ integers are chosen from this, there must be two that belong to the same group.

So now group the one hundred integers as follows:

$$\{1, 2, \ldots 20\}, \{21, 22, \ldots, 40\},$$

$$\{41, 42, \ldots, 60\},\ \{61, 62, \ldots, 80\}$$

and

$$\{81, 82, \ldots, 100\}.$$

If we select fifty five integers, we must perforce choose eleven from some group. From that group, by the above observation (let $n = 10$), there must be two that differ by 10.

Example 17.5 (1978 Putnam) Let A be any set of twenty integers chosen from the arithmetic progression $1, 4, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

Solution: We partition the thirty four elements of this progression into nineteen groups

$$\{1\}, \{52\}, \{4, 100\}, \{7, 97\}, \{10, 94\}, \ldots, \{49, 55\}.$$

Since we are choosing twenty integers and we have nineteen sets, by the Pigeonhole Principle there must be two integers that belong to one of the pairs, which add to 104.

Example 17.6 (1964 IMO) Seventeen people correspond by mail with one another—each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there at least three people who write to each other about the same topic.

Solution: Choose a particular person of the group, say Charlie. He corresponds with sixteen others. By the Pigeonhole Principle, Charlie must write to at least six of the people of one topic, say topic I. If any pair of these six people corresponds on topic I, then Charlie and this pair do the trick, and we are done. Otherwise, these six correspond amongst themselves only on topics II or III. Choose a particular person from this group of six, say Eric. By the Pigeonhole Principle, there must be three of the five remaining that correspond with Eric in one of the topics, say topic II. If amongst these three there is a pair that corresponds with each other on topic II, then Eric and this pair correspond on topic II, and we are done. Otherwise, these three people only correspond with one another on topic III, and we are done again.

Example 17.7 Given any set of ten natural numbers between 1 and 99 inclusive, prove that there are two disjoint nonempty subsets of the set with equal sums of their elements.

Solution: There are $2^{10}-1 = 1023$ non-empty subsets that one can form with a given 10-element set. To each of these subsets we associate the sum of its elements. The maximum value that any such sum can achieve is $90 + 91 + \cdots + 99 = 945 < 1023$. Therefore, there must be at least two different subsets S, T that have the same element sum. Then $S \setminus (S \cap T)$ and $T \setminus (S \cap T)$ also have the same element sum.

Example 17.8 Given any 9 integers whose prime factors lie in the set $\{3, 7, 11\}$ prove that there must be two whose product is a square.

Solution: For an integer to be a square, all the exponents of its prime factorization must be even. Any integer in the given set has a prime factorization of the form $3^a 7^b 11^c$. Now each triplet (a, b, c) has one of the following 8 parity patterns: (even, even, even), (even, even, odd), (even, odd, even), (even, odd, odd), (odd, even, even), (odd, even, odd), (odd, odd, even), (odd, odd, odd). In a group of 9 such integers, there must be two with the same parity patterns in the exponents. Take these two. Their product is a square, since the sum of each corresponding exponent will be even.

Definition 17.1 A *lattice point* (m, n) on the plane is one having integer coordinates.

Definition 17.2 The midpoint of the line joining (x, y) to (x_1, y_1) is the point

$$\left(\frac{x + x_1}{2}, \frac{y + y_1}{2}\right).$$

Example 17.9 Five lattice points are chosen at random. Prove that one can always find two so that the midpoint of the line joining them is also a lattice point.

Solution: There are four parity patterns: (even, even), (even, odd), (odd, odd), (odd, even). By the Pigeonhole Principle among five lattice points there must be two having the same parity pattern. Choose these two. It is clear that their midpoint is an integer.

Example 17.10 Prove that among $n + 1$ integers, there are always two whose difference is always divisible by n.

Solution: There are n possible different remainders when an integer is divided by n, so among $n + 1$ different integers there must be two integers in the group leaving the same remainder, and their difference is divisible by n.

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18 Proof by Contradiction

Example 18.1 Show, without using a calculator, that $6 - \sqrt{35} < \dfrac{1}{10}$.

Solution: Assume that $6 - \sqrt{35} \geq \dfrac{1}{10}$. Then $6 - \dfrac{1}{10} \geq \sqrt{35}$ or $59 \geq 10\sqrt{35}$. Squaring both sides we obtain $3481 \geq 3500$, which is clearly nonsense. Thus it must be the case that $6 - \sqrt{35} < \dfrac{1}{10}$.
In a **proof by contradiction** (or *reductio ad absurdum*), we assume, along with the hypotheses, the logical negation of the statement that we are trying to prove, and then reach some kind of contradiction. Upon reaching a contradiction, we conclude that the original assumption (i.e. the negation of the statement we are trying to prove) is false, and thus the statement that we are trying to prove must be true.

Example 18.2 Prove that $\sqrt{2}$ is irrational.

Solution #1: Assume that $\sqrt{2}$ is rational, i.e. that $\sqrt{2} = \dfrac{a}{b}$, with positive integers a, b. This yields $2b^2 = a^2$. Now both a^2 and b^2 have an even number of prime factors. So $2b^2$ has an odd number of primes in its factorization and a^2 has an even number of primes in its factorization. This is a contradiction, as we have seen that every integer has a *unique* prime factorization. Thus, $\sqrt{2}$ is irrational.

Solution #2: Assume that $\sqrt{2}$ is rational, i.e. that $\sqrt{2} = \dfrac{a}{b}$, where the fraction $\frac{a}{b}$ is in lowest terms. Then $2b^2 = a^2$, so a^2 must be even. Thus, a must be even as well (if a were odd, then a^2 would also be odd). Thus, a must be equal to 2 times an integer, so we can write $a = 2t$ for some positive integer t. Then

$$2b^2 = a^2 = (2t)^2 = 4t^2,$$

so $b^2 = 2t^2$. Thus b^2 is even, so b must be even as well. But this is a contradiction, since the fraction $\frac{a}{b}$ is in lowest terms. Thus, $\sqrt{2}$ is irrational.

Example 18.3 Prove that there are no positive integer solutions to the equation

$$x^2 - y^2 = 1.$$

Solution: Assume that there is a solution (x, y) where x and y are positive integers. Then we can factor the left-hand side of the equation to obtain

$$(x - y)(x + y) = 1.$$

Since x and y are both positive integers, $x-y$ and $x+y$ are integers. Thus, $x-y = 1$ and $x + y = 1$ or $x - y = -1$ and $x + y = -1$. In the first case, we add the two equations to obtain $x = 1$ and $y = 0$, which contradicts the assumption that x and y

are both positive. In the second case, we add the two equations to obtain $x = -1$ and $y = 0$, which is again a contradiction. Thus, there are no positive integer solutions to the equation $x^2 - y^2 = 1$.

Example 18.4 Let $a_1, a_2, \ldots, a_n$ be an arbitrary permutation of the numbers $1, 2, \ldots, n$, where n is an odd number. Prove that the product

$$(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$$

is even.

Solution: It is enough to prove that some difference $a_k - k$ is even. Assume that all the differences $a_k - k$ are odd. Clearly

$$S = (a_1 - 1) + (a_2 - 2) + \cdots + (a_n - n) = 0,$$

since the a_k's are a reordering of $1, 2, \ldots, n$. S is an odd number of summands of odd integers adding to the even integer 0. This is impossible. Our initial assumption that all the $a_k - k$ are odd is wrong, so one of these is even and hence the product is even.

Example 18.5 Let a, b be real numbers and assume that for all numbers $\epsilon > 0$ the following inequality holds:

$$a < b + \epsilon.$$

Prove that $a \leq b$.

Solution: Assume that $a > b$. Hence $\dfrac{a-b}{2} > 0$. Since the inequality $a < b + \epsilon$ holds for every $\epsilon > 0$ in particular it holds for $\epsilon = \dfrac{a-b}{2}$. This implies that

$$a < b + \frac{a-b}{2}$$

, so

$$a < b.$$

This is a contradiction, so we conclude that $a \leq b$.

Example 18.6 (Euclid) Show that there are infinitely many prime numbers.

Solution: The following beautiful proof is attributed to Euclid.

Assume that there are only finitely many (say, n) prime numbers. Then $\{p_1, p_2, \ldots, p_n\}$ is a list that exhausts all the primes. Consider the number

$$N = p_1 p_2 \cdots p_n + 1.$$

This is a positive integer, clearly greater than 1. Observe that none of the primes on the list $\{p_1, p_2, \ldots, p_n\}$ divides N, since division by any of these primes leaves a remainder of 1. Since N is larger than any of the primes on this list, it is either a prime or divisible by a prime outside this list. Thus we have shown that the assumption that any finite list of primes leads to the existence of a prime outside this list, so we have reached a contradiction. This implies that the number of primes is infinite.

Example 18.7 Let $n > 1$ be a composite integer. Prove that n has a prime factor $p \leq \sqrt{n}$.

Solution: Since n is composite, n can be written as $n = ab$ where both $a > 1, b > 1$ are integers. Now, if both $a > \sqrt{n}$ and $b > \sqrt{n}$ then $n = ab > \sqrt{n}\sqrt{n} = n$, a contradiction. Thus one of these factors must be $\leq \sqrt{n}$ and it must have a prime factor $\leq \sqrt{n}$.

Example 18.8 If a, b, c are odd integers, prove that $ax^2 + bx + c = 0$ does not have a rational number solution.

Solution: Suppose $\dfrac{p}{q}$ is a rational solution to the equation. We may assume that p and q have no prime factors in common, so either p and q are both odd, or one is odd and the other even. Now

$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0 \implies ap^2 + bpq + cq^2 = 0.$$

If both p and p were odd, then $ap^2 + bpq + cq^2$ is also odd and hence $\neq 0$. Similarly if one of them is even and the other odd then either $ap^2 + bpq$ or $bpq + cq^2$ is even and $ap^2 + bpq + cq^2$ is odd. This contradiction proves that the equation cannot have a rational root.

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19 Mathematical Induction

Mathematical induction is a powerful method for proving statements that are "indexed" by the integers. For example, induction can be used to prove the following:

- The sum of the interior angles of any n-gon is $180(n-2)$ degrees.
- The inequality $n! > 2^n$ is true for all integers $n \geq 4$.
- $7^n - 1$ is divisible by 6 for all integers $n \geq 1$.

Each assertion can be put in the form:

$$P(n) \textit{ is true for all integersn} \geq n_0,$$

where $P(n)$ is a statement involving the integer n, and n_0 is the starting point, or *base case*. For example, for the third assertion, $P(n)$ is the statement $7^n - 1$ is divisible by 6, and the base case is $n_0 = 1$. Here's how induction works:

1. First, establish the truth of $P(n_0)$. This is called the *base case*.
2. Next, *assume* that $P(n)$ is true for some integer n. This is called the **inductive hypothesis**. Then, show that the inductive hypothesis implies that the statement $P(n+1)$ must also be true.

Observe that these two steps are sufficient to prove that $P(n)$ is true for all integers $n \geq n_0$, as $P(n_0)$ is true by step (1), and step (2) then implies that $P(n_0+1)$ is true, which implies that $P(n_0+2)$ is true, etc.

You can think of induction in the following way. Suppose that you have arranged infinitely many dominos in a line, corresponding to statements $P(1)$, $P(2)$, $P(3)$, ... If you make the first domino fall, then you can be sure that all of the dominos will fall, provided that whenever one domino falls, it will knock down its neighbor. Knocking the first domino down is analogous to establishing the base case. Showing that each falling domino knocks down its neighbor is equivalent to showing that $P(n)$ implies $P(n+1)$.

Example 19.1 Prove that $n! > 2^n$ for all integers $n \geq 4$.

Solution: $P(n)$ is the statement $n! > 2^n$. The base case is $n_0 = 4$.

(i) Base case.

$$4! = 24 > 2^4 = 16,$$

so the base case $P(4)$ is true.

(ii) Inductive hypothesis. Assume that $n! > 2^n$. We must use this assumption to prove that $(n+1)! > 2^{n+1}$. The left-hand side of the inductive hypothesis is $n!$, and the left-hand side of the statement that we want to prove is $(n+1)! = (n+$

1)$n!$. Thus, it seems natural to multiply both sides of the inductive hypothesis by $(n+1)$.

$$\begin{aligned} n! &> 2^n \\ (n+1)n! &> (n+1)2^n \\ (n+1)! &> (n+1)2^n. \end{aligned}$$

Finally, note that $(n+1) > 2$, so

$$(n+1)! > (n+1)2^n > 2 \cdot 2^n > 2^{n+1},$$

so we conclude that

$$(n+1)! > 2^{n+1},$$

as needed.

Thus, $n! > 2^n$ for all integers $n \geq 4$.

Example 19.2 Prove that the expression $3^{3n+3} - 26n - 27$ is a multiple of 169 for all natural numbers n.

Solution: $P(n)$ is the assertion $3^{3n+3} - 26n - 27$ is a multiple of 169, and the base case is $n_0 = 1$.

(i) Base case. Observe that $3^{3(1)+3} - 26(1) - 27 = 676 = 4(169)$ so $P(1)$ is true.

(ii) Inductive hypothesis. Assume that $P(n)$ is true, i.e. that is, that there is an integer M such that

$$3^{3n+3} - 26n - 27 = 169M.$$

We must prove that there is an integer K so that

$$3^{3(n+1)+3} - 26(n+1) - 27 = 169K.$$

We have:

$$\begin{aligned} 3^{3(n+1)+3} - 26(n+1) - 27 &= 3^{3n+3+3} - 26n - 26 - 27 \\ &= 27(3^{3n+3}) - 26n - 27 - 26 \\ &= 27(3^{3n+3}) - 26n - 26(26n) - 27 - 26(27) + 26(26n) + 26(27) - 26 \\ &= 27(3^{3n+3} - 26n - 27) + 676n + 676 \\ &= 27(169M) + 169 \cdot 4n + 169 \cdot 4 \\ &= 169(27M + 4n + 4). \end{aligned}$$

Thus, $3^{3n+3} - 26n - 27$ is a multiple of 169 for all natural numbers n.

Example 19.3 Prove that if k is odd, then 2^{n+2} divides

$$k^{2^n} - 1$$

for all natural numbers n.

Solution: Let k be odd. $P(n)$ is the statement that 2^{n+2} is a divisor of $k^{2^n} - 1$, and the base case is $n_0 = 1$.

(i) Base case.

$$k^2 - 1 = (k-1)(k+1)$$

is divisible by 8 for any odd natural number k since $k-1$ and $k+1$ are consecutive even integers.

(ii) Inductive hypothesis. Assume that $2^{n+2}a = k^{2^n} - 1$ for some integer a. Then

$$k^{2^{n+1}} - 1 = (k^{2^n} - 1)(k^{2^n} + 1) = 2^{n+2}a(k^{2^n} + 1).$$

Since k is odd, $k^{2^n} + 1$ is even and so $k^{2^n} + 1 = 2b$ for some integer b. This gives

$$k^{2^{n+1}} - 1 = 2^{n+2}a(k^{2^n} + 1) = 2^{n+3}ab,$$

and so the assertion follows by induction.

Example 19.4 The *Fibonacci Numbers* are given by $f_0 = 1,\ f_1 = 1,\ f_{n+1} = f_n + f_{n-1}, n \geq 1$, that is every number after the second one is the sum of the preceding two.
The first several terms of the Fibonacci sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

Prove that for all integers $n \geq 1$,

$$f_{n-1}f_{n+1} = f_n^2 + (-1)^{n+1}.$$

Solution: $P(n)$ is the statement that

$$f_{n-1}f_{n+1} = f_n^2 + (-1)^{n+1}$$

and the base case is $n_0 = 1$.

(i) Base case. If $n = 1$, then $2 = f_0 f_2 = 1^2 + (-1)^2 = f_1^2 + (-1)^{1+1}$.

(ii) Inductive hypothesis. Assume that $f_{n-1}f_{n+1} = f_n^2 + (-1)^{n+1}$. Then, using the fact that $f_{n+2} = f_n + f_{n+1}$, we have

$$\begin{aligned} f_n f_{n+2} &= f_n(f_n + f_{n+1}) \\ &= f_n^2 + f_n f_{n+1} \\ &= f_{n-1}f_{n+1} - (-1)^{n+1} + f_n f_{n+1} \\ &= f_{n+1}(f_{n-1} + f_n) + (-1)^{n+2} \\ &= f_{n+1}^2 + (-1)^{n+2}, \end{aligned}$$

which establishes the assertion by induction.

Example 19.5 Prove that

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

is an integer for all integers $n \geq 0$.

Solution: $P(n)$ is the statement that

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

is an integer and the base case is $n_0 = 0$.

(i) Base case. Since 0 is an integer, the statement is clearly true when $n = 0$.

(ii) Inductive hypothesis. Assume that

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

is an integer. We must show that

$$\frac{(n+1)^5}{5} + \frac{(n+1)^4}{2} + \frac{(n+1)^3}{3} - \frac{n+1}{30}$$

is also an integer. We have:

$$\frac{(n+1)^5}{5} + \frac{(n+1)^4}{2} + \frac{(n+1)^3}{3} - \frac{n+1}{30} =$$
$$\frac{n^5+5n^4+10n^3+10n^2+5n+1}{5} + \frac{n^4+4n^3+6n^2+4n+1}{2} + \frac{n^3+3n^2+3n+1}{3} - \frac{n+1}{30}$$
$$\left[\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right] + \left[n^4 + 2n^3 + 2n^2 + n + 2n^3 + 3n^2 + 2n + n^2 + n\right],$$

which is an integer by the inductive hypothesis and since the second grouping is a sum of integers.

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